Infinite-dimensional stochastic differential equations related to random matrices

Hirofumi Osada

Address: Faculty of Mathematics, Kyushu University,
Fukuoka, 819-0395, JAPAN
Tel.: +81-92-802-4489 , Fax: +81-92-802-4489
Email: osada@math.kyushu-u.ac.jp

February 8, 2011

Abstract

We solve infinite-dimensional stochastic differential equations (ISDEs) describing an infinite number of Brownian particles interacting via two-dimensional Coulomb potentials. The equilibrium states of the associated unlabeled stochastic dynamics are the Ginibre random point field and Dyson's measures, which appear in random matrix theory. To solve the ISDEs we establish an integration by parts formula for these measures. Because the long-range effect of two-dimensional Coulomb potentials is quite strong, the properties of Brownian particles interacting with two-dimensional Coulomb potentials are remarkably different from those of Brownian particles interacting with Ruelle's class interaction potentials. As an example, we prove that the interacting Brownian particles associated with the Ginibre random point field satisfy plural ISDEs. [1] [2]

1 Introduction

Consider infinitely many Brownian particles $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ moving in $\mathbb{R}^d$ interacting via the two-dimensional (2D) Coulomb potentials Ψ_β:

$$\Psi_\beta(x) = -\beta \log |x| \qquad (\beta > 0). \tag{1.1}$$

Then the stochastic dynamics $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ is described by the following infinite-dimensional stochastic differential equation (ISDE):

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \qquad (i \in \mathbb{N}). \tag{1.2}$$

Here $\{B^i\}_{i\in\mathbb{N}}$ is a sequence of independent copies of d-dimensional Brownian motions and $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ is a continuous $(\mathbb{R}^d)^{\mathbb{N}}$-valued process.

[1]Keywords: Interacting Brownian particles Coulomb potentials Random matrices Ginibre random point field Dyson's model infinite-dimensional stochastic differential equations

[2]AMS subjectclass 2000MSC 82C22 MSC 15A52 MSC 60J60 MSC 60K35 MSC 82B21

Physically this dynamics describes the motion of an infinite system of a one-component plasma in $\mathbb{R}^d$. If $d = 2$, so that the particles can be thought of as infinitely long parallel charged lines perpendicular to the confining plane [1]. Because the Coulomb interactions Ψ_β are two-dimensional, the ISDE (1.2) is meaningful only for $d = 1, 2$.

The purpose of this paper is to solve the ISDE (1.2) by relating the system to random matrix theory. Namely, we consider the cases $d = 2$, $\beta = 2$ and $d = 1$, $\beta = 1, 2, 4$. These are related to Ginibre ensembles ($d = 2$, $\beta = 2$) and Gaussian random matrices called GOE, GUE, and GSE ($d = 1$, $\beta = 1, 2, 4$). The former is the thermodynamic limit of the distributions of eigen values of non Hermitian random Gaussian matrices, and the latter are those of orthogonal, unitary and symplectic random Gaussian matrices, respectively.

For a given interaction potential Φ, the study of ISDEs of this type was initiated by Lang [6], [7], and followed by Shiga [17], Fritz [3], Tanemura [21] and others. In these works Φ is assumed to be a Ruelle's class potential, that is, Φ is super stable and integrable at infinity. In addition, Φ is assumed to be C_0^3 ([6], [7]) or to decay exponentially at infinity. Hence, the polynomial decay potentials have been excluded even for Ruelle's category.

We develop a new approach to solve ISDEs of this kind for general potentials Φ. As an application we solve (1.2) with (d, β) as mentioned above. Our condition is easily checked for all Ruelle's class potentials with suitable smoothness outside the origin, so we give a new result even for this class.

All our conditions to solve ISDEs are stated in terms of geometric assumptions on the ISDEs. The first step is the existence of the equilibrium state of the dynamics given by the ISDE. In case of Ruelle's class potentials this step is trivial because the equilibrium states are Gibbs measures, whose existence is well established in [16], and the relationship between the candidate equilibrium states and the ISDE follows from the Dobrushin-Lanford-Ruelle equation (DLR equation).

On the other hand, when Φ is a 2D Coulomb potential, the situation is drastically changed. Because of the unboundedness at infinity of 2D Coulomb potentials, we can no longer use the method in [16] for the construction of equilibrium states, and the DLR equation becomes meaningless. In the 2D Coulomb case, even the construction of infinite-volume measures for general β has not yet been established. Moreover, the lack of the DLR equation requires a new device for clarifying the connection between the candidates for equilibrium states and the ISDE (1.2). For the construction, we use a result from random matrix theory [8] and determinantal random point fields [19], [18]. To clarify the relation between the measures and the ISDEs, we establish the integration by parts formula for the candidates for the equilibrium states. Because the candidates for the equilibrium states are given by the correlation functions defined by the determinants of some kernels, such a formula is extremely non-trivial. The calculation of such an integration by parts formula for the measures appearing in random matrix theory is the heart of the present paper.

The ISDE (1.2) with $d = \beta = 2$ is the primary example of the present paper. In this case we have plural ISDEs representing the same diffusion (see Theorems 2.1 and 2.2). Except for the unboundedness at infinity, the 2D Coulomb potentials have rather simple structure; they yield only repulsive force. The property of the associated stochastic dynamics is however drastically changed from that of the stochastic dynamics given by Ruelle's class potentials. Indeed, we will

prove in a forth coming paper that the tagged particles are sub-diffusive. This contrasts strikingly with the result of Ruelle's class potentials [11]. We conjecture that when $d = 2$, a phase transition occurs in β.

The ISDE (1.2) with $d = 1$ and $\beta = 2$ was first investigated by Spohn [20], and followed by [10], Nagao-Forrester [9], and Katori-Tanemura [5]. In these works, the dynamics was constructed by Dirichlet forms or in terms of space-time correlation functions. The ISDE was only intuitively obtained by analogy with SDEs for finite particle approximations. In this sense the ISDE (1.2) has not yet been solved. We remark that the passage of the SDE representation from the finite particle systems to the infinite one is an extremely sensitive problem because of the long range nature of the 2D Coulomb potentials.

It is plausible that our method is applicable to other measures appearing in random matrix theory and determinantal random point fields. We do not pursue this here.

The organization of the paper is as follows: In Section 2 we set up the mathematics and state some of the main theorems. In Section 3 we prove Theorems 2.6 and 2.7. These theorems give a general theory for solving ISDEs with long range potentials. In Section 4 we prove Theorem 4.5, which gives a general procedure for the integration by parts formula. In Section 5 we give a sufficient condition in (4.30), which is a key to the integration by parts formula in Section 4. In Section 6 we establish the integration by parts formula for the Ginibre random point field, which corresponds to the case $d = 2$ and $\beta = 2$ in (1.2). In Section 7 we prove Theorems 2.1–2.3. In Section 8 we prove the integration by parts formula for Dyson's models and complete the proof of Theorems 2.4 and 2.5. These theorems correspond to the cases $d = 1$ and $\beta = 1, 2, 4$ in (1.2). In the Appendix we give the definition of the determinantal kernels of the case $d = 1$ and $\beta = 1, 4$.

2 Set up and main results

Let $S = \mathbb{R}^d$ and $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\, \mathsf{s}(K) < \infty \text{ for all compact sets } K \subset S\}$, where δ_a stands for the delta measure at a. We endow S with the vague topology, under which S is a Polish space. S is called the configuration space over S. We write $\mathsf{s}(x) = \mathsf{s}(\{x\})$. Let

$$\mathsf{S}_{\text{s.i.}} = \{\mathsf{s} \in \mathsf{S}\,;\ \mathsf{s}(x) \le 1 \text{ for all } x \in S,\ \mathsf{s}(S) = \infty\}. \tag{2.1}$$

By definition, $\mathsf{S}_{\text{s.i.}}$ is the set of the configurations consisting of an infinite number of single point measures.

For an infinite or finite product S^k of S we define the map $\mathfrak{u}$ from S^k to the set of measures on S by $\mathfrak{u}((s_j)) = \sum_{j=1}^k \delta_{s_j}$. We omit k from the notation. We consider the restriction of $\mathfrak{u}$ on $\mathfrak{u}^{-1}(\mathsf{S}_{\text{s.i.}})$. Let $\mathfrak{u}_{\text{path}}$ be the map from $C([0,\infty); S^k \cap \mathfrak{u}^{-1}(\mathsf{S}_{\text{s.i.}}))$ to $C([0,\infty); \mathsf{S}_{\text{s.i.}})$ defined by

$$\mathfrak{u}_{\text{path}}(\mathbf{X}) = \{\sum_{j=1}^k \delta_{X_t^j}\}_{0 \le t < \infty}, \tag{2.2}$$

where $\mathbf{X} = \{(X_t^j)_j\}$. We set $\mathsf{X} = \mathfrak{u}_{\text{path}}(\mathbf{X})$.

A symmetric locally integrable function $\rho^n : S^n \to [0, \infty)$ is called the n-point correlation function of a probability measure μ on S w.r.t. the Lebesgue measure if ρ^n satisfies

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i) - k_i)!} d\mu \tag{2.3}$$

for any sequence of disjoint bounded measurable subsets $A_1, \ldots, A_m \subset S$ and a sequence of natural numbers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$. It is known that under a mild condition $\{\rho^n\}_{n \in \mathbb{N}}$ determine the measure μ [19].

Let μ_{gin} be the probability measure on the configuration space over $S = \mathbb{R}^2$ whose n-point correlation function ρ^n_{gin} w.r.t. the Lebesgue measure is given by

$$\rho^n_{\mathrm{gin}}(x_1, \ldots, x_n) = \det[\mathsf{K}_{\mathrm{gin}}(x_i, x_j)]_{1 \le i,j \le n}, \tag{2.4}$$

where $\mathsf{K}_{\mathrm{gin}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ is the kernel defined by

$$\mathsf{K}_{\mathrm{gin}}(x, y) = \pi^{-1} e^{-\frac{|x|^2}{2} - \frac{|y|^2}{2}} \cdot e^{x \bar{y}}. \tag{2.5}$$

Here we identify $\mathbb{R}^2$ as $\mathbb{C}$ by the obvious correspondence: $\mathbb{R}^2 \ni x = (x_1, x_2) \mapsto x_1 + \mathrm{i} x_2 \in \mathbb{C}$, and $\bar{y} = y_1 - \mathrm{i} y_2$ means the complex conjugate under this identification, where $\mathrm{i} = \sqrt{-1}$. It is known that $\mu_{\mathrm{gin}}(\mathsf{S}_{\mathrm{s.i.}}) = 1$. Moreover, μ_{gin} is translation and rotation invariant. μ_{gin} is called the Ginibre random point field.

Theorem 2.1. *There exists a set* $\mathsf{S}_{\mathrm{gin}}$ *such that*

$$\mu_{\mathrm{gin}}(\mathsf{S}_{\mathrm{gin}}) = 1, \quad \mathsf{S}_{\mathrm{gin}} \subset \mathsf{S}_{\mathrm{s.i.}}, \tag{2.6}$$

and that, for all $\mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_{\mathrm{gin}})$, *there exists an* $(\mathbb{R}^2)^{\mathbb{N}}$*-valued continuous process* $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$, *and* $(\mathbb{R}^2)^{\mathbb{N}}$*-valued Brownian motion* $\mathbf{B} = (B^i)_{i \in \mathbb{N}}$ *satisfying*

$$dX^i_t = dB^i_t + \lim_{r \to \infty} \sum_{|X^i_t - X^j_t| < r,\, j \ne i} \frac{X^i_t - X^j_t}{|X^i_t - X^j_t|^2} dt \quad (i \in \mathbb{N}), \tag{2.7}$$

$$\mathbf{X}_0 = \mathbf{s}. \tag{2.8}$$

Moreover, $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ *satisfies*

$$P(\mathbf{X}_t \in \mathfrak{u}^{-1}(\mathsf{S}_{\mathrm{gin}}),\ 0 \le \forall t < \infty) = 1, \tag{2.9}$$

$$P(\sup_{0 \le t \le u} |X^i_t| < \infty \ \textit{for all}\ u, i \in \mathbb{N}) = 1. \tag{2.10}$$

One specific aspect of the ISDE (2.7) is that its solution satisfies the second ISDE. Such a phenomenon never occurs in Ruelle's class potentials.

Theorem 2.2. *The solution* $(\mathbf{X}, \mathbf{B})$ *in Theorem 2.1 satisfies*

$$dX^i_t = dB^i_t - X^i_t dt + \lim_{r \to \infty} \sum_{|X^j_t| < r,\, j \ne i} \frac{X^i_t - X^j_t}{|X^i_t - X^j_t|^2} dt \quad (i \in \mathbb{N}). \tag{2.11}$$

To clarify the meaning of the ISDEs we define the measure μ^1_{gin} on $S \times \mathsf{S}$ by

$$\mu^1_{\text{gin}}(A \times B) = \int_A \mu_{\text{gin},x}(B)\rho^1_{\text{gin}}(x)dx, \tag{2.12}$$

where $\mu_{\text{gin},x} = \mu_{\text{gin}}(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$ is the Palm measure conditioned at x and ρ^1_{gin} is the 1-point correlation function of μ_{gin}. Let $\mathsf{b}, \tilde{\mathsf{b}} : S \times \mathsf{S} \to \mathbb{R}^2$ be such that

$$\mathsf{b}(x, \mathsf{y}) = \lim_{r \to \infty} \sum_{|x - y_i| < r} \frac{x - y_i}{|x - y_i|^2}, \tag{2.13}$$

$$\tilde{\mathsf{b}}(x, \mathsf{y}) = \lim_{r \to \infty} \sum_{|y_i| < r} \frac{x - y_i}{|x - y_i|^2}, \qquad \text{where } \mathsf{y} = \sum_i \delta_{y_i}. \tag{2.14}$$

We will see in Lemma 7.2 (3) that these two series converge in $L^2_{\text{loc}}(\mu^1_{\text{gin}})$. We remark that neither of the series converges absolutely and, as a result, $\mathsf{b} \neq \tilde{\mathsf{b}}$. Let $\mathsf{X}^{i*}_t = \sum_{j \neq i, j \in \mathbb{N}} \delta_{X^j_t}$. Then (2.7) and (2.11) can be rewritten as follows:

$$dX^i_t = dB^i_t + \mathsf{b}(X^i_t, \mathsf{X}^{i*}_t)dt \qquad (i \in \mathbb{N}), \tag{2.15}$$

$$dX^i_t = dB^i_t - X^i_t dt + \tilde{\mathsf{b}}(X^i_t, \mathsf{X}^{i*}_t)dt \qquad (i \in \mathbb{N}). \tag{2.16}$$

A diffusion with state space S_0 is a family of continuous stochastic processes with the strong Markov property starting from each point of the state space S_0. So far, the unlabeled dynamics are known to be S-valued diffusions. We refine this as follows:

Theorem 2.3. *Let $\mathbf{P}_{\mathbf{s}}$ be the distribution of the fully labeled dynamics $\mathbf{X}$ given by Theorem 2.1. Then $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_{\text{gin}})}$ is a diffusion with state space $\mathfrak{u}^{-1}(\mathsf{S}_{\text{gin}})$.*

The second example is Dyson's model. Let $S = \mathbb{R}$ and let $\mu_{\text{dys},\beta}$ $(\beta = 1, 2, 4)$ be the probability measure on S whose n-point correlation function ρ^n_β is given by

$$\rho^n_\beta(x_1, \ldots, x_n) = \det[\mathsf{K}_\beta(x_i - x_j)]_{1 \leq i,j \leq n}. \tag{2.17}$$

Here we take $\mathsf{K}_2(x) = \sin(\pi x)/\pi x$. The definition of K_β for $\beta = 1, 4$ is given in the Appendix. We use quaternions to denote the kernel K_β for $\beta = 1, 4$. The precise meaning of the determinant of (2.17) for $\beta = 1, 4$ is given by (9.3).

The kernel K_2 is called the sine kernel. We remark that $\mathsf{K}_2(t) = \frac{1}{2\pi}\int_{|k| \leq \pi} e^{\mathrm{i}kt}\, dk$ and $0 \leq \mathsf{K}_2 \leq \mathrm{Id}$ as an operator on $L^2(\mathbb{R})$.

Theorem 2.4. *Let $\beta = 1, 2, 4$. Then there exists a set $\mathsf{S}_{\text{dys},\beta}$ such that*

$$\mu_{\text{dys},\beta}(\mathsf{S}_{\text{dys},\beta}) = 1, \quad \mathsf{S}_{\text{dys},\beta} \subset \mathsf{S}_{\text{s.i.}}, \tag{2.18}$$

and that, for all $\mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_{\text{dys},\beta})$, there exists an $\mathbb{R}^{\mathbb{N}}$-valued continuous process $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$, and $\mathbb{R}^{\mathbb{N}}$-valued Brownian motion $\mathbf{B} = (B^i)_{i \in \mathbb{N}}$ satisfying

$$dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r \to \infty} \sum_{|X^i_t - X^j_t| \leq r,\, j \neq i} \frac{1}{X^i_t - X^j_t} dt \quad (i \in \mathbb{N}), \tag{2.19}$$

$$\mathbf{X}_0 = \mathbf{s}. \tag{2.20}$$

Moreover, $\mathbf{X}$ satisfies

$$P(\mathfrak{u}(\mathbf{X}_t) \in \mathsf{S}_{\text{dys},\beta},\ 0 \leq \forall t < \infty) = 1. \tag{2.21}$$

Theorem 2.5. *Let $\mathbf{S}_{\mathrm{dys},\beta} = \mathfrak{u}^{-1}(\mathsf{S}_{\mathrm{dys},\beta})$. Let $\mathbf{P}_{\mathbf{s}}$ be the distribution of $\mathbf{X}$ given by Theorem 2.4. Then $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_{\mathrm{dys},\beta}}$ is a diffusion with state space $\mathbf{S}_{\mathrm{dys},\beta}$.*

To solve the infinite-dimensional SDEs above, we prepare a general theory. Let $\sigma : S \times \mathsf{S} \to \mathbb{R}^{d^2}$ and $\mathsf{b} : S \times \mathsf{S} \to \mathbb{R}^d$ be measurable functions. Let $\mathsf{a} = \sigma\sigma^t$. We assume for each $(x, \mathsf{y}) \in S \times \mathsf{S}$

$$0 < \sum_{m,n=1}^{d} \mathsf{a}_{mn}(x, \mathsf{y})\xi_m\xi_n \le c_1|\xi|^2 \quad \text{for all } \xi = (\xi_m) \in \mathbb{R}^d\backslash\{\mathbf{0}\}. \tag{2.22}$$

Here c_1 is a positive constant independent of (x, y). For $(X^i)_{i\in\mathbb{N}}$ we set $\mathsf{X}_t^{i*} = \sum_{j\neq i, j\in\mathbb{N}} \delta_{X_t^j}$ as before. Then the ISDEs we study are of the form:

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{i*})dB_t^i + \mathsf{b}(X_t^i, \mathsf{X}_t^{i*})dt \quad (i \in \mathbb{N}). \tag{2.23}$$

Let $\check{\sigma}(x, (y_j)_{j\in\mathbb{N}})$ be the function defined on $S \times S^{\mathbb{N}}$ being symmetric in $(y_j)_{j\in\mathbb{N}}$ for each x and satisfying $\check{\sigma}(x, (y_j)_{j\in\mathbb{N}}) = \sigma(x, \sum_{j\in\mathbb{N}} \delta_{y_j})$. We set $\check{\mathsf{b}}$ similarly. Then we can rewrite (2.23) as (2.24):

$$dX_t^i = \check{\sigma}(X_t^i, (X_t^j)_{j\neq i})dB_t^i + \check{\mathsf{b}}(X_t^i, (X_t^j)_{j\neq i})dt \quad (i \in \mathbb{N}). \tag{2.24}$$

Let $\check{\mathsf{a}} = \check{\sigma}\check{\sigma}^t$. Write $\check{\mathsf{a}} = [\check{\mathsf{a}}_{kl}]_{1\le k,l\le d}$ and $\check{\mathsf{b}} = (\check{\mathsf{b}}_k)_{1\le k\le d}$. Then intuitively the generator $\mathbf{L}$ of the diffusion given by (2.24) is

$$\mathbf{L} = \frac{1}{2}\sum_{i\in\mathbb{N}}\sum_{k,l=1}^{d} \check{\mathsf{a}}_{kl}(s_i, (s_j)_{j\neq i})\frac{\partial^2}{\partial s_{ik}\partial s_{il}} + \sum_{i\in\mathbb{N}}\sum_{k=1}^{d} \check{\mathsf{b}}_k(s_i, (s_j)_{j\neq i})\frac{\partial}{\partial s_{ik}}. \tag{2.25}$$

Here $s_i = (s_{i1}, \ldots, s_{id}) \in S \equiv \mathbb{R}^d$.

Our strategy for solving ISDE (2.23) and (2.24) is to use a geometric property behind the ISDE (2.23). We first consider an invariant probability measure μ of the unlabeled dynamics associated with (2.23). Namely, we consider a probability measure μ whose log derivative d^μ satisfies $\mathsf{b}(x, \mathsf{y}) = \nabla_x\mathsf{a}(x, \mathsf{y}) + \mathsf{a}(x, \mathsf{y})\mathsf{d}^\mu(x, \mathsf{y})$. Here, to be more precise, d^μ is the log derivative of the measure μ^1 given by (2.26), and the definition of d^μ is given by (2.32).

Note that for a given pair (a, μ), b is uniquely determined. We construct the unlabeled diffusion associated with (a, μ) by using the Dirichlet space given by (a, μ) and prove that the labeled process consisting of each component of the unlabeled diffusion satisfies (2.23) and (2.24).

If there were a Dirichlet space associated with the fully labeled diffusion $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$, we could use the Ito formula for each component X^i and X^iX^j, and prove that $\mathbf{X}$ satisfies (2.25) since all coordinate functions x^i, x^ix^j $(i, j \in \mathbb{N})$ would be in the domain of the Dirichlet space locally. We emphasize that there exist no Dirichlet spaces associated with the fully labeled diffusion $\mathbf{X}$. Instead we introduce an infinite sequence of Dirichlet spaces associated with the k-labeled process $\{((X_t^1, \ldots, X_t^k, \sum_{j>k} \delta_{X_t^j}))\}$ for all $k = 0, 1, \ldots$. This sequence of k-labeled processes has consistency and satisfies the ISDEs (2.23) and (2.24).

Let μ be a probability measure on $(\mathsf{S}, \mathcal{B}(\mathsf{S}))$. Let ρ^k be the k-point correlation function of μ w.r.t. the Lebesgue measure. Let μ^k be the measure on $S^k \times \mathsf{S}$ defined by

$$\mu^k(A \times B) = \int_A \mu_{\mathbf{x}}(B)\rho^k(\mathbf{x})d\mathbf{x}. \tag{2.26}$$

Here $\mathbf{x} = (x_1, \ldots, x_k) \in S^k$ and $d\mathbf{x} = dx_1 \cdots dx_k$. Moreover $\mu_{\mathbf{x}}$ is the Palm measure conditioned at $\mathbf{x} = (x_1, \ldots, x_k)$ defined by

$$\mu_{\mathbf{x}} = \mu(\cdot - \sum_{i=1}^{k} \delta_{x_i} | \ \mathsf{s}(x_i) \geq 1 \text{ for } i = 1, \ldots, k). \tag{2.27}$$

We now introduce Dirichlet forms describing the k-labeled dynamics. For a subset $A \subset S$ we define the map $\pi_A : \mathsf{S} \to \mathsf{S}$ by $\pi_A(\mathsf{s}) = \mathsf{s}(A \cap \cdot)$. We say a function $f : \mathsf{S} \to \mathbb{R}$ is local if f is $\sigma[\pi_A]$-measurable for some compact set $A \subset S$. We say f is smooth if $\tilde{f}$ is smooth, where $\tilde{f}((s_i))$ is the permutation invariant function in (s_i) such that $f(\mathsf{s}) = \tilde{f}((s_i))$ for $\mathsf{s} = \sum_i \delta_{s_i}$.

Let $\mathcal{D}_\circ$ be the set of all local, smooth functions on S with compact support. For $f, g \in \mathcal{D}_\circ$ we set $\mathbb{D}^{\mathsf{a}}[f, g] : \mathsf{S} \to \mathbb{R}$ by

$$\mathbb{D}^{\mathsf{a}}[f, g](\mathsf{s}) = \frac{1}{2} \sum_i \sum_{m,n=1}^{d} \mathsf{a}_{mn}(s_i, \mathsf{s}_i^*) \frac{\partial \widetilde{f}(\mathbf{s})}{\partial s_{im}} \frac{\partial \widetilde{g}(\mathbf{s})}{\partial s_{in}}. \tag{2.28}$$

Here $\mathsf{s} = \sum_i \delta_{s_i}$, $\mathsf{s}_i^* = \sum_{j \neq i} \delta_{s_j}$, $s_i = (s_{i1}, \ldots, s_{id}) \in S$, and $\mathbf{s} = (s_i)$. For given f and g in $\mathcal{D}_\circ$, it is easy to see that the right-hand side of (2.28) depends only on s. So $\mathbb{D}^{\mathsf{a}}[f, g]$ is well defined. For $f, g \in C_0^\infty(S^k) \otimes \mathcal{D}_\circ$ let $\nabla^{\mathsf{a},k}[f, g]$ be the function on $S^k \times \mathsf{S}$ defined by

$$\nabla^{\mathsf{a},k}[f, g](\mathbf{x}, \mathsf{s}) = \frac{1}{2} \sum_{j=1}^{k} \sum_{m,n=1}^{d} \mathsf{a}_{mn}(x_j, \sum_{l \neq j}^{k} \delta_{x_l} + \mathsf{s}) \frac{\partial f(\mathbf{x}, \mathsf{s})}{\partial x_{jm}} \frac{\partial g(\mathbf{x}, \mathsf{s})}{\partial x_{jn}}. \tag{2.29}$$

where $\mathbf{x} = (x_j) \in S^k$ and $x_j = (x_{j1}, \ldots, x_{jd}) \in S$. We set $\mathbb{D}^{\mathsf{a},k}$ for $k \geq 1$ by

$$\mathbb{D}^{\mathsf{a},k}[f, g](\mathbf{x}, \mathsf{s}) = \nabla^{\mathsf{a},k}[f, g](\mathbf{x}, \mathsf{s}) + \mathbb{D}^{\mathsf{a}}[f(\mathbf{x}, \cdot), g(\mathbf{x}, \cdot)](\mathsf{s}). \tag{2.30}$$

Let $(\mathcal{E}^{\mathsf{a},k}, C_0^\infty(S^k) \otimes \mathcal{D}_\circ)$ be the bilinear form defined by

$$\mathcal{E}^{\mathsf{a},k}(f, g) = \int_{S^k \times \mathsf{S}} \mathbb{D}^{\mathsf{a},k}[f, g] d\mu^k. \tag{2.31}$$

When $k = 0$, we take $\mathbb{D}^{\mathsf{a},0} = \mathbb{D}^{\mathsf{a}}$, $\mu^0 = \mu$, and $\mathcal{E}^{\mathsf{a},0} = \mathcal{E}^{\mathsf{a}}$. We set $L^2(\mu) = L^2(\mathsf{S}, \mu)$ and $L^2(\mu^k) = L^2(S^k \times \mathsf{S}, \mu^k)$ and so on.

We assume that there exists a probability measure μ on S with correlation functions $\{\rho^k\}_{k \in \mathbb{N}}$ satisfying (A.1)–(A.5):
(A.1) ρ^k is locally bounded for each $k \in \mathbb{N}$.
(A.2) There exists a $\mathsf{d}^\mu = (\mathsf{d}_m^\mu)_{m=1,\ldots,d} \in \{L^1_{\mathrm{loc}}(\mu^1)\}^d$ such that

$$\int_{S \times \mathsf{S}} \mathsf{d}^\mu f d\mu^1 = -\int_{S \times \mathsf{S}} \nabla_x f d\mu^1 \quad \text{for all } f \in C_0^\infty(S) \otimes \mathcal{D}_\circ. \tag{2.32}$$

Moreover, d^μ satisfies

$$\mathsf{b} = \frac{1}{2} \{\nabla_x \mathsf{a}\} \mathsf{d}^\mu + \frac{1}{2} \mathsf{a} \mathsf{d}^\mu, \quad \mathsf{b} \in L^2_{\mathrm{loc}}(\mu^1). \tag{2.33}$$

Here $\nabla_x f = (\frac{\partial f(x,\mathsf{s})}{\partial x_m})_{m=1,\ldots,d}$ and $\nabla_x \mathsf{a} = [\frac{\partial \mathsf{a}_{mn}(x,\mathsf{s})}{\partial x_n}]_{m,n=1,\ldots,d}$, where $x = (x_m)$.
(A.3) $(\mathcal{E}^{\mathsf{a},k}, C_0^\infty(S^k) \otimes \mathcal{D}_\circ)$ is closable on $L^2(\mu^k)$ for each $k \in \{0\} \cup \mathbb{N}$.

(A.4) $\mathrm{Cap}^{\mu}(\{\mathsf{S}_{\mathrm{s.i.}}\}^c) = 0$.
(A.5) There exists a $T > 0$ such that for each $R > 0$

$$\liminf_{r\to\infty} (\{\int_{|x|\le r+R} \rho^1(x)dx\}\{\int_{\frac{r}{\sqrt{(r+R)T}}}^{\infty} e^{-u^2/2}du\}) = 0. \tag{2.34}$$

Let $(\mathcal{E}^{\mathsf{a},k}, \mathcal{D}^{\mathsf{a},k})$ be the closure of $(\mathcal{E}^{\mathsf{a},k}, C_0^{\infty}(S^k)\otimes\mathcal{D}_{\circ})$ on $L^2(\mu^k)$. It is known [14, Lemma 2.3] that $(\mathcal{E}^{\mathsf{a},k}, \mathcal{D}^{\mathsf{a},k})$ is quasi-regular and that the associated diffusion $(\mathsf{P}^k, \mathsf{X}^k)$ exists. These diffusions have consistency in the sense of (3.6) and (3.7) (see [14]). We remark that Cap^{μ} in (A.4) is the capacity of the Dirichlet space $(\mathcal{E}^{\mathsf{a},0}, \mathcal{D}^{\mathsf{a},0}, L^2(\mu))$. We call d^{μ} the log derivative of μ.

The assumptions (A.4) and (A.5) have clear dynamical interpretations. Indeed, (A.4) means that particles never collide each other. Moreover, (A.5) means that each labeled particle never explodes [14].

Theorem 2.6. *Assume* (A.1)–(A.5). *Then there exists an* S_0 *such that*

$$\mu(\mathsf{S}_0) = 1, \quad \mathsf{S}_0 \subset \mathsf{S}_{\mathrm{s.i.}}, \tag{2.35}$$

and that, for all $\mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, *there exists an* $S^{\mathbb{N}}$*-valued continuous process* $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$, *and* $(\mathbb{R}^d)^{\mathbb{N}}$*-valued Brownian motion* $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ *satisfying*

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{i*})dB_t^i + \mathsf{b}(X_t^i, \mathsf{X}_t^{i*})dt \quad (i \in \mathbb{N}), \tag{2.36}$$
$$\mathbf{X}_0 = \mathbf{s}. \tag{2.37}$$

Moreover, $\mathbf{X}$ *satisfies*

$$P(\mathfrak{u}(\mathbf{X}_t) \in \mathsf{S}_0,\ 0 \le \forall t < \infty) = 1. \tag{2.38}$$

Remark 2.1. Let $(\mathsf{P}^1, \mathsf{X}^1)$ be the diffusion associated with $(\mathcal{E}^{\mathsf{a},1}, \mathcal{D}^{\mathsf{a},1}, L^2(\mu^1))$. Let $N = \{N_t\}$ be the additive functional defined by $N_t = \int_0^t \mathsf{b}(\mathsf{X}_u^1)du$. The assumption $\mathsf{b} \in L^2_{\mathrm{loc}}(\mu^1)$ in (2.33) is used to ensure that $N = \{N_t\}$ is a continuous additive functional locally of zero energy in the sense of [2], and that $N = \{N_t\}$ possesses an increasing sequence of open sets $\{O_n\}$ in $S \times \mathsf{S}$ such that $\mathrm{Cap}^{\mu^1}(\cup_n O_n^c) = 0$ and that

$$\lim_{t\to 0} \frac{1}{t} \int_{S\times\mathsf{S}} \mathrm{E}^1_{(x,\mathsf{y})}[N_t]\varphi(x,\mathsf{y})d\mu^1 = \int_{S\times\mathsf{S}} \mathsf{b}(x,\mathsf{y})\varphi(x,\mathsf{y})d\mu^1 \tag{2.39}$$

for any $\varphi \in \mathcal{D}^{\mathsf{a},1}$ such that $\varphi = 0$ on O_n^c. Here $\mathrm{E}^1_{(x,\mathsf{y})}$ denotes the expectation w.r.t. the diffusion measure starting at (x,y). Indeed, the property $\mathsf{b} \in L^2_{\mathrm{loc}}(\mu^1)$ is used only for this. So we can relax the assumption that $\mathsf{b} \in L^2_{\mathrm{loc}}(\mu^1)$. This fact will be used for Dyson's model with $\beta = 1$ because $\mathsf{b} \in L^p_{\mathrm{loc}}(\mu^1)$ for any $1 \le p < 2$, but $\mathsf{b} \notin L^2_{\mathrm{loc}}(\mu^1)$ in this case.

Theorem 2.7. *Let* $\mathbf{S}_0$ *be the subset of* $S^{\mathbb{N}}$ *defined by* $\mathbf{S}_0 = \mathfrak{u}^{-1}(\mathsf{S}_0)$. *Let* $\mathbf{P}_{\mathbf{s}}$ *be the distribution of* $\mathbf{X}$ *given by Theorem 2.6. Then* $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_0}$ *is a diffusion with state space* $\mathbf{S}_0$.

Remark 2.2. (1) There exist no Dirichlet spaces associated with the fully labeled diffusion $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_0}$ because the diffusion $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_0}$ has no invariant measures.

Hence Theorem 2.7 does not follow directly from the Dirichlet form theory.
(2) The solutions obtained in [3], [6], [7], [17], [21] for Ruelle's class interaction potentials are strong solutions in the sense that they are functionals of given Brownian motions. The strong Markov property of the solutions was however not proved in these works except [3]. It is an interesting open problem to prove that the solutions in Theorems 2.1 and 2.4 are strong solutions.

Example 2.1. Let Ψ be a Ruelle's class potential, smooth outside the origin. Then the associated translation invariant grand canonical Gibbs measures constructed in [16] satisfy (A.1)–(A.3) and (A.5). (A.4) is satisfied if $d \ge 2$, or $d = 1$ with Φ sufficiently repulsive at the origin [4]. More concrete examples are:
(1) Let $\Phi_{6,12}(x) = c_2\{|x|^{-12} - |x|^{-6}\}$, where $d = 3$ and $c_2 > 0$ is a constant. $\Phi_{6,12}$ is called the Lennard-Jones 6-12 potential. The corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c_2}{2}\sum_{j=1,j\neq i}^{\infty}\{\frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}}\}dt \quad (i \in \mathbb{N}).$$

(2) Let $a > d$ and set $\Phi_a(x) = (c_3/a)|x|^{-a}$, where $c_3 > 0$. Then the corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c_3}{2}\sum_{j=1,j\neq i}^{\infty}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}}dt \quad (i \in \mathbb{N}). \tag{2.40}$$

At first glance the ISDE (2.40) resembles (1.2) because (1.2) corresponds to the case $a = 0$ in (2.40). The sums in the drift terms however converge absolutely, unlike in (1.2). We emphasize that the structures of the dynamics given by the solutions of (2.40) and (1.2) are completely different from each other.

3 Proof of Theorems 2.6 and 2.7.

In this section we prove Theorems 2.6 and 2.7. We assume (A.1)–(A.5) throughout this section. Let $(\mathcal{E}^{\mathsf{a},k}, \mathcal{D}^{\mathsf{a},k})$ be the closure of $(\mathcal{E}^{\mathsf{a},k}, C_0^\infty(S^k)\otimes\mathcal{D}_\circ)$ on $L^2(\mu^k)$. We set $\mathsf{X}^k = (X^k, \mathsf{X}) \in C([0,\infty); S^k \times \mathsf{S})$.

Lemma 3.1. *Assume* (A.1) *and* (A.3)*. Then the following holds:*
(1) $(\mathcal{E}^{\mathsf{a},k}, \mathcal{D}^{\mathsf{a},k})$ *is a quasi-regular Dirichlet form on* $L^2(\mu^k)$*.*
(2) *There exists a diffusion* $\mathsf{P}^k = (\{\mathsf{P}^k_{(x,\mathsf{s})}\}_{(x,\mathsf{s})\in S^k\times\mathsf{S}}, \mathsf{X}^k)$ *associated with the Dirichlet space* $(\mathcal{E}^{\mathsf{a},k}, \mathcal{D}^{\mathsf{a},k}, L^2(\mu^k))$*.*

Proof. (1) follows from Lemma 2.3 in [14]. (2) follows from (1) and Dirichlet form theory. □

Let $\mathfrak{l}\colon \mathsf{S}_{\text{s.i.}} \to S^{\mathbb{N}}$ be a measurable map such that $\mathfrak{u}\circ\mathfrak{l}$ is the identity map. We represent this map by $\mathfrak{l}(\mathsf{s}) = (s_1, \ldots,)$, where $\mathsf{s} = \sum_{i=1}^\infty \delta_{s_i}$. The map $\mathfrak{l}$ means the label of the originally unlabeled particle s and is called a *label*. So there are infinitely many labels $\mathfrak{l}$ satisfying the above mentioned condition. Moreover, it is easy to see that $\mathfrak{u}^{-1}(\mathsf{S}_{\text{s.i.}}) = \cup_{\mathfrak{l}}\mathfrak{l}(\mathsf{S}_{\text{s.i.}})$, where the union is taken over all labels.

Let $\mathsf{S}_{\text{s.i.}}^k$ be the subset of $S^k \times \mathsf{S}$ defined by $\mathsf{S}_{\text{s.i.}}^k = \mathfrak{u}^{-1}(\mathsf{S}_{\text{s.i.}})$. For a given label $\mathfrak{l}$ as above let $\mathfrak{l}_k : \mathsf{S}_{\text{s.i.}} \to \mathsf{S}_{\text{s.i.}}^k$ be the map defined by

$$\mathfrak{l}_k(\sum_{i=1}^{\infty} \delta_{s_i}) = (s_1, \dots, s_k, \sum_{i=k+1}^{\infty} \delta_{s_i}). \tag{3.1}$$

Note that $\mathfrak{u} \circ \mathfrak{l}_k$ is the identity map.

One can extend $\mathfrak{l}$ naturally as the map from $C([0,\infty); \mathsf{S}_{\text{s.i.}})$ to $C([0,\infty); S^{\mathbb{N}})$. Indeed, for a path $\mathsf{X} = \{\mathsf{X}_t\} \in C([0,\infty); \mathsf{S}_{\text{s.i.}})$, there exists a unique $\{(X_t^i)\} \in C([0,\infty); S^{\mathbb{N}})$ such that $(X_0^i) = \mathfrak{l}(\mathsf{X}_0)$ and that $\sum_i \delta_{X_t^i} = \mathsf{X}_t$ for all $t \in [0,\infty)$. We write this map as $\mathfrak{l}_{\text{path}}(\mathsf{X}) = \{(X_t^i)\}$. We set $\mathfrak{l}_{k,\text{path}} : C([0,\infty); \mathsf{S}_{\text{s.i.}}) \to C([0,\infty); \mathsf{S}_{\text{s.i.}}^k)$ similarly as $\mathfrak{l}_{\text{path}}$ for $k \geq 1$.

We write $\mathsf{P}_{\mathsf{s}} = \mathsf{P}_{\mathsf{s}}^0$, where $\mathsf{P}_{\mathsf{s}}^0$ is given by Lemma 3.1.

Lemma 3.2. *Assume* (A.1)–(A.5). *Then there exists a set* $\tilde{\mathsf{S}}$ *satisfying*

$$\tilde{\mathsf{S}} \subset \mathsf{S}_{\text{s.i.}}, \tag{3.2}$$

$$\mathrm{Cap}^{\mu}(\tilde{\mathsf{S}}^c) = 0, \tag{3.3}$$

$$\mathsf{P}_{\mathsf{s}}(\mathsf{X}_t \in \tilde{\mathsf{S}} \textit{ for all } t) = 1 \quad \textit{for all } \mathsf{s} \in \tilde{\mathsf{S}}, \tag{3.4}$$

$$\mathsf{P}_{\mathsf{s}}(\sup_{0 \leq t \leq u} |X_t^i| < \infty \textit{ for all } u, i \in \mathbb{N}) = 1 \quad \textit{for all } \mathsf{s} \in \tilde{\mathsf{S}}. \tag{3.5}$$

Here $\mathsf{X}_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}$. *Moreover, for all* $k \in \mathbb{N}$ *and any label* $\mathfrak{l}$

$$\mathsf{P}_{\mathsf{s}^k}^k = \mathsf{P}_{\mathfrak{u}(\mathsf{s}^k)} \circ \mathfrak{l}_{k,\text{path}}^{-1} \quad \textit{for all } \mathsf{s}^k \in \mathfrak{l}_k(\tilde{\mathsf{S}}), \tag{3.6}$$

$$\mathsf{P}_{\mathsf{s}} = \mathsf{P}_{\mathfrak{l}_k(\mathsf{s})}^k \circ \mathfrak{u}_{\text{path}}^{-1} \quad \textit{for all } \mathsf{s} \in \tilde{\mathsf{S}}. \tag{3.7}$$

Proof. This lemma is immediate from Theorems 2.4 and 2.5 in [14]. □

For $\mathbf{s} \in \mathfrak{u}^{-1}(\tilde{\mathsf{S}})$ such that $\mathfrak{u}(\mathbf{s}) = \mathsf{s}$ let $\mathbf{P}_{\mathbf{s}} = \mathsf{P}_{\mathsf{s}} \circ \mathfrak{l}_{\text{path}}^{-1}$, where $\mathfrak{l}$ is a label such that $\mathfrak{l}(\mathsf{s}) = \mathbf{s}$. Let

$$C([0,\infty); \mathsf{S}_{\text{s.i.}})_{\mathsf{s}} = \{\mathsf{X} \in C([0,\infty); \mathsf{S}_{\text{s.i.}}); \mathsf{X}_0 = \mathsf{s}\}.$$

We remark that $\mathfrak{l}_{\text{path}}|_{C([0,\infty);\mathsf{S}_{\text{s.i.}})_{\mathsf{s}}} = \hat{\mathfrak{l}}_{\text{path}}|_{C([0,\infty);\mathsf{S}_{\text{s.i.}})_{\mathsf{s}}}$ for any labels $\mathfrak{l}$ and $\hat{\mathfrak{l}}$ satisfying $\mathfrak{l}(\mathsf{s}) = \hat{\mathfrak{l}}(\mathsf{s}) = \mathbf{s}$, and that $\mathfrak{u}^{-1}(\tilde{\mathsf{S}}) = \cup_{\mathfrak{l}} \mathfrak{l}(\mathsf{S}_{\text{s.i.}})$. Hence $\mathbf{P}_{\mathbf{s}}$ is well defined.

Lemma 3.3. $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s} \in \mathfrak{u}^{-1}(\tilde{\mathsf{S}})}$ *is a diffusion with state space* $\mathfrak{u}^{-1}(\tilde{\mathsf{S}})$.

Proof. We recall that $\{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s} \in \tilde{\mathsf{S}}}$ is a diffusion with state space $\tilde{\mathsf{S}}$ by Lemma 3.1 and Lemma 3.2. Since $\mathbf{P}_{\mathbf{s}}(\mathfrak{l}_{\text{path}}(C([0,\infty); \mathsf{S}_{\text{s.i.}})_{\mathsf{s}})) = 1$ and

$$\mathfrak{l}_{\text{path}}|_{C([0,\infty);\mathsf{S}_{\text{s.i.}})_{\mathsf{s}}} = \hat{\mathfrak{l}}_{\text{path}}|_{C([0,\infty);\mathsf{S}_{\text{s.i.}})_{\mathsf{s}}}$$

for any labels $\mathfrak{l}$ and $\hat{\mathfrak{l}}$ satisfying $\mathfrak{l}(\mathsf{s}) = \hat{\mathfrak{l}}(\mathsf{s}) = \mathbf{s}$, we deduce that $\mathbf{P}_{\mathbf{s}}$ depends only on P_{s} and the value of the label $\mathfrak{l}$ at s. Hence the strong Markov property follows from that of $\{\mathsf{P}_{\mathsf{s}}\}$. The continuity of the sample paths is clear by construction. □

Let $\mathsf{a} = [\mathsf{a}_{mn}]$ and b be as in (2.22) and (2.23), respectively.

Lemma 3.4. *Let $M_t^i = X_t^i - X_0^i - \int_0^t \mathsf{b}(X_u^i, \mathsf{X}_u^{i*})du$. Then there exists a $\mathsf{S}_0 \subset \tilde{\mathsf{S}}$ satisfying* $\mathrm{Cap}^\mu(\tilde{\mathsf{S}} \backslash \mathsf{S}_0) = 0$ *such that, for each $\mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, the collection of the processes $\{M^i\}_{i\in\mathbb{N}}$ under $\mathbf{P}_\mathsf{s}$ is a sequence of d-dimensional continuous local martingales such that*

$$\langle M^i, M^j\rangle_t = 0 \quad (i \neq j), \qquad \langle M^i, M^i\rangle_t = \int_0^t \mathsf{a}(X_u^i, \mathsf{X}_u^{i*})du. \tag{3.8}$$

Proof. For a diffusion process $(P, \{X_t\})$ with state space S and a continuous function f on S we write $A_t^{[f]} = f(X_t) - f(X_0)$. Then $A^{[f]}$ becomes an additive functional (AF). An AF of this type is called a Dirichlet process. It is worthwhile to note that one can apply the Fukushima decomposition for Dirichlet processes if f is locally in the domain of the Dirichlet form associated with the diffusion. We note that $A^{[f]}$ is defined as $A_t^{[f]} = \tilde{f}(X_t) - \tilde{f}(X_0)$, where $\tilde{f}$ is a quasi-continuous version of f if f is not necessary continuous but is in the domain of Dirichlet spaces.

The process $X_t^i - X_0^0$ is an AF of the unlabeled diffusion (P, X). However, $X_t^i - X_0^0$ is not a Dirichlet process of (P, X). Indeed, we can not identify the position of the ith particle without tracing all of the trajectory of the unlabeled process $\mathsf{X} = \{\mathsf{X}_u\}$ until $u \le t$. On the other hand, one can regard $X_t^i - X_0^0$ as a Dirichlet process of the labeled process $(\mathbf{P}, \mathbf{X})$ since the coordinate function x^i is a function of the state space $S^\mathbb{N}$ of $(\mathbf{P}, \mathbf{X})$. However, there is no Dirichlet form associated with the labeled process $(\mathbf{P}, \mathbf{X})$. Taking these into account, we consider the k-labeled process $((X_t^1, \ldots, X_t^k, \sum_{l=k+1}^\infty \delta_{X_t^l}))$. Here k is taken such that $i, j \le k$. We note that the k-labeled process is associated with the Dirichlet space $(\mathcal{E}^{\mathsf{a},k}, \mathcal{D}^{\mathsf{a},k}, L^2(\mu^k))$.

Applying [2, Theorem 5.5.1] to the function $x^i = x^i \otimes 1 \in \mathbb{R}^d$ and taking Lemma 3.2 into account we deduce that there exists a set $\mathsf{S}_0^k \subset \tilde{\mathsf{S}}$ satisfying $\mathrm{Cap}^\mu(\tilde{\mathsf{S}} \backslash \mathsf{S}_0^k) = 0$ and, for each $\mathsf{s} \in \mathsf{S}_0^k$, the d-dimensional AF $A^{[x^i]} = \{X_t^i - X_0^i\}$ can be decomposed under $\mathsf{P}_{\mathfrak{l}_k(\mathsf{s})}^k$ as

$$A^{[x^i]} = M^{[x^i]} + N^{[x^i]}. \tag{3.9}$$

Here $M^{[x^i]}$ is a martingale AF (MAF), locally of finite energy, and $N^{[x^i]}$ is a continuous AF (CAF) locally of zero energy. By a straightforward calculation we deduce that for any $\varphi \in C_0^\infty(S^k) \otimes \mathcal{D}_\circ$

$$-\mathcal{E}^{\mathsf{a},k}(x^i, \varphi) = \int_{S^k \times \mathsf{S}} \mathsf{b}(x^i, \sum_{l\neq i}^k \delta_{x^l} + \mathsf{y})\varphi(\mathbf{x}, \mathsf{y})d\mu^k. \tag{3.10}$$

Here $\mathbf{x} = (x^1, \ldots, x^k) \in S^k$. By $\mathsf{b} \in L^2_{\mathrm{loc}}(\mu^1)$ we see that $\mathsf{b}(x^i, \sum_{l\neq i}^k \delta_{x^l} + \mathsf{y}) \in L^2_{\mathrm{loc}}(\mu^k)$. So, by [2, Theorem 5.2.4] together with localization, we deduce that

$$N_t^{[x^i]} = \int_0^t \mathsf{b}(X_u^i, \sum_{l\neq i}^k \delta_{X_u^l} + \sum_{l=k+1}^\infty \delta_{X_u^l})du. \tag{3.11}$$

Hence $M^{[x^i]} = A^{[x^i]} - N^{[x^i]} = M^i$ under $\mathsf{P}_{\mathfrak{l}_k(\mathsf{s})}^k$. This, combined with the relation $(M^i, \mathsf{P}_{\mathfrak{l}_k(\mathsf{s})}^k) = (M^i, \mathbf{P}_\mathsf{s})$ given by Lemma 3.2, yields that $(M^i, \mathbf{P}_\mathsf{s})$ is a continuous

local martingale. As for the quadratic variation of M^i, we note that

$$\mathbb{D}^{\mathsf{a},k}[x_m^i, x_n^j](\mathbf{x}, \mathsf{y}) = \begin{cases} 0 & (i \neq j) \\ \frac{1}{2}\mathsf{a}_{mn}(x^i, \sum_{l\neq i}^k \delta_{x^l} + \mathsf{y}) & (i = j) \end{cases}. \tag{3.12}$$

Here $x^i = (x_m^i) \in \mathbb{R}^d$. Since

$$2\mathbb{D}^{\mathsf{a},k}[x_m^i x_n^j f, x_m^i x_n^j] - \mathbb{D}^{\mathsf{a},k}[(x_m^i x_n^j)^2, f] = \mathbb{D}^{\mathsf{a},k}[x_m^i, x_n^j]\, f \tag{3.13}$$

and $\mathcal{E}^{\mathsf{a},k}(f,g) = \int \mathbb{D}^{\mathsf{a},k}[f,g]d\mu^k$, we deduce (3.8) from (3.12) and [2, Theorem 5.2.3].

Let $\mathsf{S}_0 = \cap_{k=1}^{\infty} \mathsf{S}_0^k$. Then by (3.3) and $\mathrm{Cap}^{\mu}(\tilde{\mathsf{S}}\backslash\mathsf{S}_0^k) = 0$ ($\forall k$) we deduce that $\mathrm{Cap}^{\mu}(\tilde{\mathsf{S}}\backslash\mathsf{S}_0) = 0$. Hence S_0 satisfies the requirement of Lemma 3.4. $\square$

Proof of Theorem 2.6. For $\mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$ let $\mathbf{P_s}$ as in Lemma 3.4. Let $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ be defined by

$$B_t^i = \int_0^t \sigma^{-1}(X_u^i, \mathsf{X}_u^{i*})dM_u^i. \tag{3.14}$$

Then B^i are d-dimensional continuous local martingales. By (3.8) and (3.14) we deduce that $[\langle B^i, B^j\rangle_t]_{i,j\in\mathbb{N}} = tE$. Here E is the unit matrix on $(\mathbb{R}^d)^{\mathbb{N}}$. We deduce that $\{B^i\}_{i\in\mathbb{N}}$ are independent copies of d-dimensional Brownian motions. Hence $(\mathbf{X}, \mathbf{B})$ under $\mathbf{P_s}$ is a solution of (2.36) and (2.37). (2.9) follows from $\mathrm{Cap}^{\mu}(\tilde{\mathsf{S}}\backslash\mathsf{S}_0) = 0$. The last statement follows from Lemma 3.2, Lemma 3.3 and $\mathrm{Cap}^{\mu}(\tilde{\mathsf{S}}\backslash\mathsf{S}_0) = 0$. $\square$

Proof of Theorem 2.7. By Lemma 3.3 we see that $\{\mathbf{P_s}\}_{\mathbf{s}\in\mathfrak{u}^{-1}(\tilde{\mathsf{S}})}$ is a diffusion with state space $\mathfrak{u}^{-1}(\tilde{\mathsf{S}})$. By Lemma 3.4 the set S_0 satisfies $\mathsf{S}_0 \subset \tilde{\mathsf{S}}$ and $\mathrm{Cap}^{\mu}(\tilde{\mathsf{S}}\backslash\mathsf{S}_0) = 0$. Hence we deduce that $\mathbf{P_s}(\mathbf{X}_t \in \mathfrak{u}^{-1}(\mathsf{S}_0) \text{ for all } t) = 1$ for each $\mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$. So we conclude that $\{\mathbf{P_s}\}_{\mathbf{s}\in\mathfrak{u}^{-1}(\mathsf{S}_0)}$ is a diffusion with state space $\mathfrak{u}^{-1}(\mathsf{S}_0)$. $\square$

4 Log derivative of random point fields.

Let μ be a probability measure on S with locally bounded n-point correlation function ρ^n for each $n \in \mathbb{N}$. Let μ^1 be the measure defined by (2.26) with $k = 1$. In this section we present a sufficient condition for the existence of the log derivative d^{μ} in $L^p_{\mathrm{loc}}(\mu^1)$ with $1 < p$ (Theorem 4.3) and its explicit representation (Theorem 4.5). We shall apply these to the Ginibre random point field and Dyson's model in the subsequent sections.

We set $S_r = \{x \in S\,; |s| < r\}$. Let $\{\mu^N\}$ be a sequence of probability measures on S. We assume that their n-point correlation functions $\{\rho^{N,n}\}$ satisfy for each $r \in \mathbb{N}$

$$\lim_{N\to\infty} \rho^{N,n}(\mathbf{x}) = \rho^n(\mathbf{x}) \quad \text{uniformly on } S_r^n, \tag{4.1}$$

$$\sup_{N\in\mathbb{N}} \sup_{\mathbf{x}\in S_r^n} \rho^{N,n}(\mathbf{x}) \leq c_4^{-n} n^{c_5 n}, \tag{4.2}$$

where $0 < c_4(r) < \infty$ and $0 < c_5(r) < 1$ are constants independent of $n \in \mathbb{N}$.

Let $\sigma_r^{N,n}$ be the n-density function of μ^N on S_r, where $r \in \mathbb{N} \cup \{\infty\}$. Then

$$\sigma_r^{N,n}(\mathbf{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{S_r^m} \rho^{N,n+m}(\mathbf{x}, \mathbf{y}) d\mathbf{y}. \tag{4.3}$$

Let σ_r^n be the n-density function of μ on S_r. Then the same equality as (4.3) holds. By (4.1)–(4.3) we deduce for each $r \in \mathbb{N}$ that

$$\lim_{N\to\infty} \sigma_r^{N,n}(\mathbf{x}) = \sigma_r^n(\mathbf{x}) \quad \text{uniformly on } S_r^n \text{ for all } n \in \mathbb{N}. \tag{4.4}$$

We remark that (4.2) and (4.4) imply $\{\mu^N\}_{N\in\mathbb{N}}$ converge weakly to μ.

Let μ_x^N be the Palm measure conditioned at x as before. Let $\rho_x^{N,n}$ (resp. $\sigma_{x,r}^{N,n}$) be the n-point correlation (resp. density) function of μ_x^N. Let $\mu^{N,1}$ be the measure defined by (2.26) with $n = 1$. Then we deduce that

$$\sigma_{x,r}^{N,m}(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{S_r^n} \rho_x^{N,m+n}(\mathbf{x}, \mathbf{y}) d\mathbf{y}, \tag{4.5}$$

$$\int f d\mu^{N,1} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{S\times S_r^n} \tilde{f}_n(x, \mathbf{y}) \rho^{N,1}(x) \sigma_{x,r}^{N,n}(\mathbf{y}) dx d\mathbf{y}. \tag{4.6}$$

Here $f \in C_0^\infty(S) \otimes \mathcal{D}_\circ$ and $f(x, \cdot)$ is $\sigma[\pi_{S_r}]$-measurable for each $x \in S$. Moreover, $\tilde{f}_n(x, \mathbf{y})$ is the function on $S \times S_r^n$ being symmetric in $\mathbf{y} = (y_1, \ldots, y_n)$ for each x and $f(x, \mathsf{y}) = \tilde{f}_n(x, \mathbf{y})$ when $\mathsf{y}(S_r) = n$ and $\mathsf{y} = \sum_{i=1}^n \delta_{y_i}$. We set $d\mathbf{y} = dy_1 \cdots dy_n$. It is easy to see that

$$\rho_x^{N,n}(\mathbf{y}) = \rho^{N,1+n}(x, \mathbf{y})/\rho^{N,1}(x), \quad \rho_x^n(\mathbf{y}) = \rho^{1+n}(x, \mathbf{y})/\rho^1(x). \tag{4.7}$$

Here ρ_x^n is the n-point correlation function of μ_x.

Lemma 4.1. *Let $\sigma_{x,r}^n$ be the n-density function of μ_x on S_r. Then for all n, r, s*

$$\lim_{N\to\infty} \rho^{N,1}(x) \sigma_{x,r+s}^{N,n}(\mathbf{y}) = \rho^1(x) \sigma_{x,r+s}^n(\mathbf{y}) \quad \textit{uniformly on } S_r \times S_{r+s}^n, \tag{4.8}$$

$$\lim_{N\to\infty} \int f d\mu^{N,1} = \int f d\mu^1 \quad \textit{for any } f \in C_0(S \times \mathsf{S}), \tag{4.9}$$

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu^{N,1}(\{(x, \mathsf{y}) \in S_r \times \mathsf{S}\,;\, \mathsf{y}(S_{r+s}) \geq n\}) = 0, \tag{4.10}$$

$$\lim_{n\to\infty} \mu^1(\{(x, \mathsf{y}) \in S_r \times \mathsf{S}\,;\, \mathsf{y}(S_{r+s}) \geq n\}) = 0. \tag{4.11}$$

Proof. Combining (4.1), (4.2), (4.5), and (4.7) implies (4.8). (4.9) follows from (4.6) and (4.8). (4.10) and (4.11) are clear by (4.1), (4.7), and the assumption that ρ^n are locally bounded. □

Let $\mathcal{B}(S_r)$ be the Borel σ-field of S_r. We regard $\mathcal{B}(S_r)$ as a subset of $\mathcal{B}(S)$ in an obvious manner and denote it by the same symbol $\mathcal{B}(S_r)$. Let $\varpi_s : S \times \mathsf{S} \to S \times \mathsf{S}$ such that $\varpi_s(x, \mathsf{y}) = (x, \sum_{|x-y_i|<s} \delta_{y_i})$, where $\mathsf{y} = \sum_i \delta_{y_i}$. Let

$$\mathcal{F}_{r,s} = \{\mathcal{B}(S_r) \times \mathcal{B}(\mathsf{S})\} \cap \sigma[\varpi_s].$$

Set $c_6(r, N) = \mu^{N,1}(S_r \times \mathsf{S})$. Then by (4.2) $\sup_N c_6(r, N) < \infty$ for each $r \in \mathbb{N}$. Without loss of generality, we can assume that $c_6 > 0$ for all r, N. So let $\bar{\mu}_r^{N,1}$ be the probability measure defined by $\bar{\mu}_r^{N,1}(\cdot) = \mu^{N,1}(\cdot \cap S_r \times \mathsf{S})/c_6$.

We assume that each μ^N has a log derivative $\mathsf{d}^N = \mathsf{d}^N(x,\mathsf{y})$ such that $\mathsf{d}^N - u^N \in L^p_{\mathrm{loc}}(\mu^{N,1})$ for some $1 < p < \infty$, where $u^N = u^N(x)$ is a distribution on S. We note that u^N is supposed to be independent of $\mathsf{y} \in \mathsf{S}$. Let $\bar{\mathsf{d}}^N_s \in L^p_{\mathrm{loc}}(\mu^{N,1})$ be such that for all $r \in \mathbb{N}$

$$\begin{aligned} 1_{S_r}\bar{\mathsf{d}}^N_s &= \mathrm{E}^{\bar{\mu}^{N,1}_r}[\mathsf{d}^N - u^N|\mathcal{F}_{r,s}] && \text{for a.s. } \bar{\mu}^{N,1}_r \\ &= \mathrm{E}^{\bar{\mu}^{N,1}_r}[\mathsf{d}^N|\mathcal{F}_{r,s}] - 1_{S_r}u^N && \text{for a.s. } \bar{\mu}^{N,1}_r. \end{aligned} \tag{4.12}$$

Then $\{1_{S_r}\bar{\mathsf{d}}^N_s\}_{s\in\mathbb{N}}$ is a $\{\mathcal{F}_{r,s}\}$-martingale w.r.t. $\bar{\mu}^{N,1}_r$ for each r. We remark that the second equality in (4.12) comes from the fact that u^N is independent of y.

Lemma 4.2. *Let $1 < p < \hat{p} < \infty$. Assume* (4.1) *and* (4.2)*. Assume that*

$$c_7 := \limsup_{N\to\infty} \int_{S_r\times\mathsf{S}} |\mathsf{d}^N - u^N|^{\hat{p}} d\mu^{N,1} < \infty \quad \textit{for all } r \in \mathbb{N}, \tag{4.13}$$

where $c_7(r)$ depends only on r. Assume that there exists a $u\colon S\to\mathbb{R}^d$ satisfying

$$\lim_{N\to\infty} u^N = u \quad \textit{in } L^{\hat{p}}_{\mathrm{loc}}(S,dx). \tag{4.14}$$

Then there exists a subsequence of $\{\{\bar{\mathsf{d}}^N_s\}_{s\in\mathbb{N}}\}_N$, denoted by the same symbol, with limit $\{\bar{\mathsf{d}}_s\}_{s\in\mathbb{N}}$ satisfying the following: For all $s \in \mathbb{N}$ and $\mathcal{F}_{r,s}$-measurable $\varphi \in C_0^\infty(S)\otimes\mathcal{D}_\circ$

$$\int_{S_r\times\mathsf{S}} \bar{\mathsf{d}}_s\varphi\, d\mu^1 = \lim_{N\to\infty}\int_{S_r\times\mathsf{S}} \bar{\mathsf{d}}^N_s\varphi\, d\mu^{N,1}, \tag{4.15}$$

$$\int_{S_r\times\mathsf{S}} |\bar{\mathsf{d}}_s|^p\, d\mu^1 \le \liminf_{N\to\infty}\int_{S_r\times\mathsf{S}} |\bar{\mathsf{d}}^N_s|^p\, d\mu^{N,1} \le c_7^{p/\hat{p}}\mu^1(S_r\times\mathsf{S})^{(\hat{p}-p)/\hat{p}}. \tag{4.16}$$

Moreover, $\bar{\mathsf{d}} := \lim_{s\to\infty}\bar{\mathsf{d}}_s$ converges in $L^p_{\mathrm{loc}}(\mu^1)$ and μ^1-almost everywhere.

Proof. By (4.12) we see that $\int_{S_r\times\mathsf{S}}|\bar{\mathsf{d}}^N_s|^{\hat{p}}d\mu^{N,1} \le \int_{S_r\times\mathsf{S}}|\mathsf{d}^N - u^N|^{\hat{p}}d\mu^{N,1}$. Hence by (4.13) we deduce that

$$\limsup_{N\to\infty}\sup_{s\in\mathbb{N}}\int_{S_r\times\mathsf{S}} |\bar{\mathsf{d}}^N_s|^{\hat{p}}d\mu^{N,1} \le c_7 \quad \text{for each } r. \tag{4.17}$$

For $(x,\mathsf{y}) \in S_r\times\mathsf{S}$ we write $\mathsf{y} = \sum_i \delta_{y_i}$ and $\mathbf{y} = (y_i)$. We set

$$\mathsf{S}^{N,m}_t = \{(x,\mathsf{y}) \in S_r\times\mathsf{S}\,;\, \rho^{N,1}(x)\sigma^{N,m}_{x,r+s}(\mathbf{y}) < a_t,\ \mathsf{y}(S_{r+s}) = m\} \tag{4.18}$$

and S^m_t similarly as $\mathsf{S}^{N,n}_t$ by replacing $\rho^{N,1}(x)\sigma^{N,n}_{x,r+s}(\mathbf{y})$ by $\rho^1(x)\sigma^m_{x,r+s}(\mathbf{y})$. Here $\{a_t\}_{t\in\mathbb{N}}$ is an increasing sequence of positive numbers such that $\lim_{t\to\infty}a_t = \infty$ and that for each $m,r,s,t,\in\mathbb{N}$

$$\mu^1(\{(x,\mathsf{y}) \in S_r\times\mathsf{S}\,;\, \rho^1(x)\sigma^m_{x,r+s}(\mathbf{y}) = a_t\}) = 0. \tag{4.19}$$

We set $\mathsf{T}^{N,n}_t = \bigcup_{m=1}^n \mathsf{S}^{N,m}_t$ and $\mathsf{T}^n_t = \bigcup_{m=1}^n \mathsf{S}^m_t$. By (4.8), (4.17), (4.18), and

the fact that $1_{S_r}\bar{\mathsf{d}}_s^N$ are $\mathcal{B}(S_r)\times\sigma[\pi_{S_{r+s}}]$-measurable we see that

$$\begin{aligned}
&\lim_{N\to\infty}\Big|\int_{\mathsf{T}_t^{N,n}}|\bar{\mathsf{d}}_s^N|^p d\mu^{N,1}-\int_{\mathsf{T}_t^{N,n}}|\bar{\mathsf{d}}_s^N|^p d\mu^1\Big| &(4.20)\\
\le&\lim_{N\to\infty}\Big\{\sup_{1\le m\le n}\sup_{\mathsf{S}_t^{N,m}}\frac{|\rho^{N,1}(x)\sigma_{x,r+s}^{N,m}(\mathbf{y})-\rho^1(x)\sigma_{x,r+s}^m(\mathbf{y})|}{\rho^{N,1}(x)\sigma_{x,r+s}^{N,m}(\mathbf{y})}\Big\}\int_{\mathsf{T}_t^{N,n}}|\bar{\mathsf{d}}_s^N|^p d\mu^{N,1}\\
\le&\lim_{N\to\infty}\Big\{\sup_{1\le m\le n}\sup_{\mathsf{S}_t^{N,m}}\frac{|\rho^{N,1}(x)\sigma_{x,r+s}^{N,m}(\mathbf{y})-\rho^1(x)\sigma_{x,r+s}^m(\mathbf{y})|}{a_t}\Big\}c_7^{p/\hat{p}}\mu^{N,1}(S_r\times\mathsf{S})^{(\hat{p}-p)/\hat{p}}\\
=&\ 0. &(4.21)
\end{aligned}$$

By applying the Hölder inequality to $|\bar{\mathsf{d}}_s^N|^p$ and by using (4.17) we have

$$\int_{S_r\times\mathsf{S}\backslash\mathsf{T}_t^{N,n}}|\bar{\mathsf{d}}_s^N|^p d\mu^{N,1}\le c_7^{p/\hat{p}}\mu^{N,1}(S_r\times\mathsf{S}\backslash\mathsf{T}_t^{N,n})^{(\hat{p}-p)/\hat{p}}. \tag{4.22}$$

By (4.8), (4.10), (4.19), and $\lim_{t\to\infty}a_t=\infty$ we deduce that

$$\lim_{n\to\infty}\lim_{t\to\infty}\limsup_{N\to\infty}\mu^{N,1}(S_r\times\mathsf{S}\backslash\mathsf{T}_t^{N,n})\le\lim_{n\to\infty}\lim_{t\to\infty}\mu^1(S_r\times\mathsf{S}\backslash\mathsf{T}_t^n)=0. \tag{4.23}$$

Combining (4.22) and (4.23) we obtain

$$\lim_{n\to\infty}\lim_{t\to\infty}\limsup_{N\to\infty}\int_{S_r\times\mathsf{S}\backslash\mathsf{T}_t^{N,n}}|\bar{\mathsf{d}}_s^N|^p d\mu^{N,1}=0. \tag{4.24}$$

By (4.17), (4.20), and (4.24) we obtain

$$\lim_{n\to\infty}\lim_{t\to\infty}\limsup_{N\to\infty}\int_{\mathsf{T}_t^{N,n}}|\bar{\mathsf{d}}_s^N|^p d\mu^1\le c_7^{p/\hat{p}}\mu^1(S_r\times\mathsf{S})^{(\hat{p}-p)/\hat{p}}. \tag{4.25}$$

By (4.25) we can choose a subsequence of $\{\bar{\mathsf{d}}_s^N\}$, denoted by the same symbol, such that $\{1_{\mathsf{T}_t^{N,n}}\bar{\mathsf{d}}_s^N\}_{N\in\mathbb{N}}$ converge weakly in $L^p(S_r\times\mathsf{S},\mu^1)$ to $\bar{\mathsf{d}}_s^{t,n}$ for each $s,t,n\in\mathbb{N}$. We can take the subsequence in such a way that the limit points $\{\bar{\mathsf{d}}_s^{t,n}\}$ satisfy

$$1_{\mathsf{T}_t^n}\bar{\mathsf{d}}_s^{t,n}=1_{\mathsf{T}_t^n}\bar{\mathsf{d}}_s^{t',n'}\ \text{ for any } n\le n' \text{ and } t\le t'.$$

Hence we can rewrite the limit points as $\bar{\mathsf{d}}_s^{t,n}=1_{\mathsf{T}_t^n}\bar{\mathsf{d}}_s$ for any $s,t,n\in\mathbb{N}$. By construction $\lim_{n\to\infty}\lim_{t\to\infty}\bar{\mathsf{d}}_s^{t,n}=\bar{\mathsf{d}}_s$ μ^1-a.s.. Then by Fatou's lemma and (4.25) we obtain the first inequality in (4.16). The second one is immediate from the Hölder inequality.

By (4.8) with the cut off argument similar to (4.20)–(4.25) and the fact that $1_{S_r}\bar{\mathsf{d}}_s^N\varphi$ are $\mathcal{B}(S_r)\times\sigma[\pi_{S_{r+s}}]$-measurable we obtain (4.15).

Let $\bar{\mu}_r^1=\mu^1(\cdot\cap S_r\times\mathsf{S})/c_8$, where $c_8=\mu^1(S_r\times\mathsf{S})$. By (4.12) and (4.15) we deduce that $\{1_{S_r}\bar{\mathsf{d}}_s\}_{s\in\mathbb{N}}$ is a martingale w.r.t. $\bar{\mu}_r^1$ for each $r\in\mathbb{N}$. Then the last claim follows from the martingale convergence theorems and (4.16). □

Theorem 4.3. *Assume the same conditions as in Lemma 4.2. Let $\bar{\mathsf{d}}$ be as in Lemma 4.2. Then the log derivative d^μ of μ exists in $L^p_{\mathrm{loc}}(\mu^1)$ and is given by $\mathsf{d}^\mu=u+\bar{\mathsf{d}}$.*

By taking $u^N = u = 0$ in Theorem 4.3 we see the following:

Corollary 4.4. *Let $1 < p < \hat{p} < \infty$. Assume* (4.1) *and* (4.2)*. Suppose*

$$\limsup_{N\to\infty} \int_{S_r \times \mathsf{S}} |\mathsf{d}^N|^{\hat{p}} d\mu^{N,1} < \infty \quad \text{for all } r \in \mathbb{N}. \tag{4.26}$$

Then the log derivative d^μ of μ exists in $L^p_{\mathrm{loc}}(\mu^1)$.

Proof. Let $\varphi \in C_0^\infty(S) \otimes \mathcal{D}_\circ$. Assume, without loss of generality, that φ is $\mathcal{F}_{r,s}$-measurable and $\varphi(x, \mathsf{y}) = 0$ for $x \notin S_r$ for some r and $s \in \mathbb{N}$.

By (4.9) in Lemma 4.1 we see that $\int \nabla\varphi d\mu^1 = \lim_{N\to\infty} \int \nabla\varphi d\mu^{N,1}$. By definition, we have $-\int \nabla\varphi \, d\mu^{N,1} = \int \mathsf{d}^N \varphi \, d\mu^{N,1}$. Hence we deduce that

$$\begin{aligned} -\int \nabla\varphi \, d\mu^1 &= \lim_{N\to\infty} \int \mathsf{d}^N \varphi \, d\mu^{N,1} \\ &= \lim_{N\to\infty} \int \{u^N + \bar{\mathsf{d}}^N_s\} \varphi \, d\mu^{N,1} && \text{by (4.12)} \\ &= \int \{u + \bar{\mathsf{d}}_s\} \varphi \, d\mu^1 && \text{by (4.14), (4.15)} \\ &= \int \{u + \bar{\mathsf{d}}\} \varphi \, d\mu^1 && \text{by Lemma 4.2,} \end{aligned}$$

which completes the proof. □

Let g, g^N, v, $v^N : S^2 \to \mathbb{R}^d$ and $w : S \to \mathbb{R}^d$ be measurable functions. We set

$$\begin{aligned} \mathsf{g}_s(x, \mathsf{y}) &= \int_{|x-y|<s} v(x, y) dy + \sum_{|x-y_i|<s} g(x, y_i), \\ \mathsf{g}^N_s(x, \mathsf{y}) &= \int_{|x-y|<s} v^N(x, y) dy + \sum_{|x-y_i|<s} g^N(x, y_i), \\ w^N_s(x, \mathsf{y}) &= \int_{s \le |x-y|} v^N(x, y) dy + \sum_{s \le |x-y_i|} g^N(x, y_i), \end{aligned} \tag{4.27}$$

where $\mathsf{y} = \sum_i \delta_{y_i}$. We assume that

$$\mathsf{d}^N(x, \mathsf{y}) = u^N(x) + \mathsf{g}^N_s(x, \mathsf{y}) + w^N_s(x, \mathsf{y}), \tag{4.28}$$

$$\lim_{N\to\infty} \mathsf{g}^N_s = \mathsf{g}_s \quad \text{in } L^{\hat{p}}_{\mathrm{loc}}(\mu^1) \quad \text{for all } s, \tag{4.29}$$

$$\lim_{s\to\infty} \limsup_{N\to\infty} \int_{S_r \times \mathsf{S}} |w^N_s(x, \mathsf{y}) - w(x)|^{\hat{p}} d\mu^{N,1} = 0, \quad w \in L^{\hat{p}}_{\mathrm{loc}}(S, dx). \tag{4.30}$$

Theorem 4.5. *Let $1 < p < \hat{p} < \infty$. Assume* (4.1), (4.2)*, and* (4.14)*. Assume* (4.28)–(4.30)*. Then the log derivative d^μ exists in $L^p_{\mathrm{loc}}(\mu^1)$ and is given by*

$$\mathsf{d}^\mu(x, \mathsf{y}) = u(x) + \lim_{s\to\infty} \mathsf{g}_s(x, \mathsf{y}) + w(x). \tag{4.31}$$

The convergence $\lim \mathsf{g}_s$ takes place in $L^p_{\mathrm{loc}}(\mu^1)$.

Proof. By (4.27) and (4.28) we see that $\mathsf{d}^N - u^N = \mathsf{g}^N_s + w^N_s$. Then (4.13) follows from (4.29) and (4.30). So all the assumptions in Lemma 4.2 are satisfied.

Hence we set $\bar{\mathsf{d}}$, $\bar{\mathsf{d}}_s^N$ and $\bar{\mathsf{d}}_s$ as in Lemma 4.2. By Theorem 4.3 the log derivative $\mathsf{d}^\mu = u + \bar{\mathsf{d}}$ exists in $L^p_{\mathrm{loc}}(\mu^1)$. We will prove that $\bar{\mathsf{d}} = \lim_{s\to\infty} \mathsf{g}_s + w$.

Let $\bar{w}_s^N$ and $\bar{w}_{ss}^N$ be functions such that for all r

$$1_{S_r}\bar{w}_s^N = 1_{S_r}\mathrm{E}^{\bar{\mu}_r^{N,1}}[w_0^N|\mathcal{F}_{r,s}], \quad 1_{S_r}\bar{w}_{ss}^N = 1_{S_r}\mathrm{E}^{\bar{\mu}_r^{N,1}}[w_s^N|\mathcal{F}_{r,s}]. \tag{4.32}$$

Then $\{1_{S_r}\bar{w}_s^N\}_{s\in\mathbb{N}}$ is a martingale w.r.t. $\bar{\mu}_r^{N,1}$ for all r. By the second equality in (4.32) combined with (4.12) and $\mathsf{d}^N = u^N + \mathsf{g}_s^N + w_s^N$ we obtain

$$\bar{\mathsf{d}}_s^N - \mathsf{g}_s^N - w = \bar{w}_{ss}^N - w. \tag{4.33}$$

By Lemma 4.2 we see that $\{\bar{\mathsf{d}}_s^N\}_N$ converge weakly to $\bar{\mathsf{d}}_s^N$ in $L^p_{\mathrm{loc}}(\mu^1)$. Hence we deduce that

$$\int_{S_r\times\mathsf{S}} |\bar{\mathsf{d}}_s - \mathsf{g}_s - w|^p d\mu^1 \le \liminf_{N\to\infty} \int_{S_r\times\mathsf{S}} |\bar{\mathsf{d}}_s^N - \mathsf{g}_s - w|^p d\mu^1. \tag{4.34}$$

By the cut off argument similar to (4.20)–(4.25) we deduce that

$$\begin{aligned}
&\limsup_{N\to\infty} \int_{S_r\times\mathsf{S}} |\bar{\mathsf{d}}_s^N - \mathsf{g}_s - w|^p d\mu^1 && (4.35)\\
&\le \lim_{n\to\infty}\lim_{t\to\infty}\limsup_{N\to\infty} \int_{\mathsf{T}_t^{N,n}} |\bar{\mathsf{d}}_s^N - \mathsf{g}_s - w|^p d\mu^{N,1} \\
&\le \limsup_{N\to\infty} \int_{S_r\times\mathsf{S}} |\bar{\mathsf{d}}_s^N - \mathsf{g}_s - w|^p d\mu^{N,1} && \text{by } \mathsf{T}_t^{N,n} \subset S_r\times\mathsf{S}\\
&= \limsup_{N\to\infty} \int_{S_r\times\mathsf{S}} |\bar{\mathsf{d}}_s^N - \mathsf{g}_s^N - w|^p d\mu^{N,1} && \text{by (4.29)}\\
&= \limsup_{N\to\infty} \int_{S_r\times\mathsf{S}} |\bar{w}_{ss}^N - w|^p d\mu^{N,1} && \text{by (4.33).}
\end{aligned}$$

By (4.30) and (4.32) together with the assumption that $w = w(x)$ is independent of y we easily see that

$$\lim_{s\to\infty}\limsup_{N\to\infty} \int_{S_r\times\mathsf{S}} |\bar{w}_{ss}^N - w|^p d\mu^{N,1} = 0. \tag{4.36}$$

Putting (4.34), (4.35) and (4.36) together we obtain

$$\lim_{s\to\infty} \int_{S_r\times\mathsf{S}} |\bar{\mathsf{d}}_s - \mathsf{g}_s - w|^p d\mu^1 = 0.$$

This combined with $\mathsf{d}^\mu = u + \bar{\mathsf{d}} = u + \lim_s \bar{\mathsf{d}}_s$ implies (4.31). Because the convergence of $\lim \bar{\mathsf{d}}_s$ takes place in $L^p_{\mathrm{loc}}(\mu^1)$, so does the convergence of $\lim \mathsf{g}_s$. $\square$

5 Sufficient conditions for (4.30)

The purpose of this section is to give sufficient conditions for $\mathsf{g}_s \in L^2_{\mathrm{loc}}(\mu^1)$ and (4.30) in terms of correlation functions.

Lemma 5.1. *Let* $g_s(x,y) = 1_{S_s}(x-y)g(x,y)$*. Then* $\mathsf{g}_s \in L^2_{\mathrm{loc}}(\mu^1)$ *follows from*

$$\int_{S_r} |\int_{|x-y|<s} v(x,y)dy|^2 \rho^1(x)dx + \int_{S_r\times S} |g_s(x,y)|^2 \rho^2(x,y)dxdy \tag{5.1}$$
$$+ \int_{S_r\times \S^2} g_s(x,y)\cdot g_s(x,z)\rho^3(x,y,z)dxdydz < \infty \quad \textit{for all } r \in \mathbb{N}.$$

Here $\cdot$ *denotes the standard inner product of* $\mathbb{R}^d$*.*

Proof. By definition $\mathsf{g}_s(x,\mathsf{y}) = \int_{|x-y|<s} v(x,y)dy + \sum_{|x-y_i|<s} g(x,y_i)$. By $d\mu^1 = \rho^1(x)\mu_x dx$, (4.7), and a simple calculation of correlation functions we see that

$$\int_{S_r\times \mathsf{S}} |\sum_{|x-y_i|<s} g(x,y_i)|^2 d\mu^1 = \int_{S_r} \mathrm{E}^{\mu_x}[|\sum_{|x-y_i|<s} g(x,y_i)|^2]\rho^1(x)dx$$
$$= \int_{S_r} \{\int_{S^2} g_s(x,y)\cdot g_s(x,z)\rho_x^2(y,z)dydz + \int_S |g_s(x,y)|^2\rho_x^1(y)dy\}\rho^1(x)dx$$
$$= \int_{S_r\times \S^2} g_s(x,y)\cdot g_s(x,z)\rho^3(x,y,z)dxdydz + \int_{S_r\times S} |g_s(x,y)|^2\rho^2(x,y)dxdy.$$

Hence (5.1) implies $\mathsf{g}_s \in L^2_{\mathrm{loc}}(\mu^1)$. □

Let $w_s^N(x,\mathsf{y})$ be as in (4.27). Let μ^N and μ_x^N be as in Section 4 with n-point correlation functions $\rho^{N,n}$ and $\rho_x^{N,n}$, respectively.

Lemma 5.2. (4.30) *with* $\hat{p} = 2$ *follows from the following:*

$$\lim_{s\to\infty}\limsup_{N\to\infty}\sup_{x\in S_r}\left|\mathrm{E}^{\mu_x^N}[w_s^N(x,\mathsf{y})] - \mathrm{E}^{\mu^N}[w_s^N(x,\mathsf{y})]\right| = 0, \tag{5.2}$$

$$\lim_{s\to\infty}\limsup_{N\to\infty}\sup_{x\in S_r}\left|\mathrm{Var}^{\mu_x^N}[w_s^N(x,\mathsf{y})] - \mathrm{Var}^{\mu^N}[w_s^N(x,\mathsf{y})]\right| = 0, \tag{5.3}$$

$$\lim_{s\to\infty}\limsup_{N\to\infty}\sup_{x\in S_r}\left|\mathrm{E}^{\mu^N}[w_s^N(x,\mathsf{y})] - w(x)\right| = 0, \tag{5.4}$$

$$\lim_{s\to\infty}\limsup_{N\to\infty}\sup_{x\in S_r}\mathrm{Var}^{\mu^N}[w_s^N(x,\mathsf{y})] = 0. \tag{5.5}$$

Proof. By (2.26) we have

$$\int_{S_r\times\mathsf{S}} |w_s^N(x,\mathsf{y})|^2 d\mu^{N,1} = \int_{S_r} \mathrm{E}^{\mu_x^N}[|w_s^N(x,\mathsf{y})|^2]\,\rho^{N,1}(x)\,dx \tag{5.6}$$
$$= \int_{S_r} \{|\mathrm{E}^{\mu_x^N}[w_s^N(x,\mathsf{y})]|^2 + \mathrm{Var}^{\mu_x^N}[w_s^N(x,\mathsf{y})]\}\,\rho^{N,1}(x)\,dx.$$

By (4.2) we see that $\{\rho^{N,1}\}_N$ is uniformly bounded. So (4.30) with $\hat{p} = 2$ follows from (5.2)–(5.6). □

We give a sufficient condition of (5.2)–(5.5) in terms of correlation functions.

Lemma 5.3. *We set* $S_{s\infty}^{x} = \{y \in S; s \le |x-y| < \infty\}$*. Then* (5.2)–(5.5) *follow from* (5.7)–(5.10).

$$\lim_{s\to\infty} \limsup_{N\to\infty} \sup_{x\in S_r} \Big| \int_{S_{s\infty}^{x}} g^N(x,y)\{\rho_x^{N,1}(y) - \rho^{N,1}(y)\}dy\Big| = 0, \tag{5.7}$$

$$\begin{aligned} \lim_{s\to\infty} \limsup_{N\to\infty} \sup_{x\in S_r} \Big| &\int_{S_{s\infty}^{x}} |g^N(x,y)|^2\{\rho_x^{N,1}(y) - \rho^{N,1}(y)\}dy \\ &- \int_{(S_{s\infty}^{x})^2} g^N(x,y)\cdot g^N(x,z)\{\rho_x^{N,2}(y,z) - \rho^{N,2}(y,z)\}dydz\Big| = 0, \end{aligned} \tag{5.8}$$

$$\lim_{s\to\infty} \limsup_{N\to\infty} \sup_{x\in S_r} \Big| \int_{S_{s\infty}^{x}} \{v^N(x,y) + g^N(x,y)\rho^{N,1}(y)\}dy - w(x)\Big| = 0, \tag{5.9}$$

$$\begin{aligned} \lim_{s\to\infty} \limsup_{N\to\infty} \sup_{x\in S_r} \Big| &\int_{S_{s\infty}^{x}} |g^N(x,y)|^2\rho^{N,1}(y)dy \\ &- \int_{(S_{s\infty}^{x})^2} g^N(x,y)\cdot g^N(x,z)\rho^{N,2}(y,z)dydz\Big| = 0. \end{aligned} \tag{5.10}$$

Proof. This lemma is clear from the standard calculation of correlation functions combined with (4.27). □

6 Log derivative of the Ginibre random point field.

In this section we calculate the log derivative $\mathsf{d}^{\mu_{\mathrm{gin}}}$ of the Ginibre random point field μ_{gin}. Let μ_{gin}^N be the probability measure on S whose n-point correlation function $\rho_{\mathrm{gin}}^{N,n}$ is given by

$$\rho_{\mathrm{gin}}^{N,n}(\mathbf{x}_n) = \det[\mathsf{K}_{\mathrm{gin}}^N(x_i,x_j)]_{1\le i,j\le n}. \tag{6.1}$$

Here $\mathbf{x}_n = (x_1,\dots,x_n)$ and $\mathsf{K}_{\mathrm{gin}}^N$ is the kernel defined by

$$\mathsf{K}_{\mathrm{gin}}^N(x,y) = \frac{1}{\pi}e^{-(|x|^2+|y|^2)/2}\{\sum_{n=0}^{N-1}\frac{(x\bar{y})^n}{n!}\}. \tag{6.2}$$

We easily see that

$$|\mathsf{K}_{\mathrm{gin}}^N(x,y)| \le \frac{1}{\pi}e^{-||x|-|y||/2}. \tag{6.3}$$

By (6.1) and (6.2) the 1-point correlation function $\rho_{\mathrm{gin}}^{N,1}$ is given by

$$\rho_{\mathrm{gin}}^{N,1}(x) = \frac{1}{\pi}e^{-|x|^2}\{\sum_{k=0}^{N-1}\frac{|x|^{2k}}{k!}\}. \tag{6.4}$$

Moreover, it holds that $\rho_{\mathrm{gin}}^{N,n} = 0$ if $n \ge N+1$, and that for $2 \le n \le N$

$$\rho_{\mathrm{gin}}^{N,n}(\mathbf{x}_n) = \frac{1}{\pi^n}\{\prod_{k=0}^{n-1}\frac{1}{k!}\}\, e^{-\sum_{k=1}^{n}|x_k|^2}\prod_{i<j}^{n}|x_i-x_j|^2. \tag{6.5}$$

Note that $\mu_{\mathrm{gin}}^N(\{\mathsf{s}(S)=N\})=1$. So by (6.5) with $n=N$ we deduce that

$$\mathsf{d}^{\mu_{\mathrm{gin}}^N}(x,\mathsf{y})=-2x+\sum_{i=1}^{N-1}\frac{2(x-y_i)}{|x-y_i|^2}\qquad (\mathsf{y}=\sum_{i=1}^{N-1}\delta_{y_i}). \tag{6.6}$$

Let μ_{gin}^1 be the measure defined by (2.26) for μ_{gin}.

Theorem 6.1. *The log derivative* $\mathsf{d}^{\mu_{\mathrm{gin}}}\in L^p_{\mathrm{loc}}(\mu_{\mathrm{gin}}^1)$ *exists for any* $1\le p<2$ *and is given by*

$$\mathsf{d}^{\mu_{\mathrm{gin}}}(x,\mathsf{y})=\lim_{r\to\infty}\sum_{|x-y_i|<r}\frac{2(x-y_i)}{|x-y_i|^2}\qquad (\mathsf{y}=\sum_{i}\delta_{y_i}). \tag{6.7}$$

The convergence of the series in the right-hand side takes place in $L^p_{\mathrm{loc}}(\mu_{\mathrm{gin}}^1)$.

To prove Theorem 6.1 we use Theorem 4.5. So we check all the conditions in Theorem 4.5. For this purpose we first prepare several lemmas.

Lemma 6.2. (4.1) *and* (4.2) *hold.* (4.14), (4.28)*, and* (4.29) *hold by taking*

$$u^N(x)=u(x)=-2x,\quad v^N(x,y)=v(x,y)=0,\quad w(x)=2x,$$
$$g^N(x,y)=g(x,y)=2(x-y)/|x-y|^2,\quad \hat{p}=2.$$

Proof. (4.1) follows immediately from (2.4), (2.5), (6.1), and (6.2). Let v_i be the norm of the ith row vector of the matrix $[\mathsf{K}_{\mathrm{gin}}^N(x_i,x_j)]_{1\le i,j\le n}$. Then by (6.3) we deduce that $v_i\le\sqrt{n}/\pi$. So we deduce from (6.1) that $\rho_{\mathrm{gin}}^{N,n}(\mathbf{x}_n)\le\prod_{i=1}^n v_i\le(\sqrt{n}/\pi)^n$, which implies (4.2). (4.14), (4.28), and (4.29) are trivial. ◻

By Lemma 6.2 it only remains to prove (4.30) with $\hat{p}=2$ for Theorem 6.1. By the argument in Section 5 we see that (4.30) follows from (5.2)–(5.5), which we will check below.

It is known that the Palm measure conditioned at x of determinantal random point fields with kernel K is again a determinantal random point field with kernel $K_x(y,z)=K(y,z)-\{K(y,x)K(x,z)/K(x,x)\}$ (see [18, Theorem 1.7]). Applying this to $\mu_{\mathrm{gin}}^{N,1}$ we deduce that the kernel $\mathsf{K}_{\mathrm{gin},x}^N$ of the Palm measure $\mu_{\mathrm{gin},x}^N$ is then given by

$$\mathsf{K}_{\mathrm{gin},x}^N(y,z)=\mathsf{K}_{\mathrm{gin}}^N(y,z)-\frac{\mathsf{K}_{\mathrm{gin}}^N(y,x)\mathsf{K}_{\mathrm{gin}}^N(x,z)}{\mathsf{K}_{\mathrm{gin}}^N(x,x)}. \tag{6.8}$$

Let $c_9=(1/\pi)\sup_{x\in S_r}e^{5|x|^2}$. Then by (6.2), (6.3) and (6.8) we deduce that

$$|\mathsf{K}_{\mathrm{gin},x}^N(y,z)-\mathsf{K}_{\mathrm{gin}}^N(y,z)|=|\frac{\mathsf{K}_{\mathrm{gin}}^N(y,x)\mathsf{K}_{\mathrm{gin}}^N(x,z)}{\mathsf{K}_{\mathrm{gin}}^N(x,x)}|\le c_9e^{-(|y|^2+|z|^2)/8}. \tag{6.9}$$

Lemma 6.3. (5.2) *and* (5.3) *hold.*

Proof. Since μ_{gin}^N and $\mu_{\mathrm{gin},x}^N$ are determinantal random point fields with kernels $\mathsf{K}_{\mathrm{gin}}^N$ and $\mathsf{K}_{\mathrm{gin},x}^N$ respectively, their 1-point correlation functions $\rho_{\mathrm{gin}}^{N,1}$ and $\rho_{\mathrm{gin},x}^{N,1}$ are given by $\rho_{\mathrm{gin}}^{N,1}(y)=\mathsf{K}_{\mathrm{gin}}^N(y,y)$ and $\rho_{\mathrm{gin},x}^{N,1}(y)=\mathsf{K}_{\mathrm{gin},x}^N(y,y)$. Moreover,

$$\rho_{\mathrm{gin}}^{N,2}(y,z)=\mathsf{K}_{\mathrm{gin}}^N(y,y)\mathsf{K}_{\mathrm{gin}}^N(z,z)-\mathsf{K}_{\mathrm{gin}}^N(y,z)\mathsf{K}_{\mathrm{gin}}^N(z,y),$$
$$\rho_{\mathrm{gin},x}^{N,2}(y,z)=\mathsf{K}_{\mathrm{gin},x}^N(y,y)\mathsf{K}_{\mathrm{gin},x}^N(z,z)-\mathsf{K}_{\mathrm{gin},x}^N(y,z)\mathsf{K}_{\mathrm{gin},x}^N(z,y).$$

Hence (5.7) and (5.8) follow from (6.9) and $g^N(x,y) = 2(x-y)/|x-y|^2$, which implies (5.2) and (5.3). $\square$

Lemma 6.4. *Set $S_{s\infty} = \{s \le |y| < \infty\}$. Suppose $r < s$. Then*

$$\int_{S_{s\infty}} \frac{2(x-y)}{|x-y|^2} \rho_{\text{gin}}^{N,1}(y)dy = 0 \quad \text{for } x \in S_r. \tag{6.10}$$

Proof. We regard $x, y \in \mathbb{R}^2$ as $x, y \in \mathbb{C}$ and $\bar{\cdot}$ denotes the complex conjugate. Then $(x-y)/|x-y|^2 = 1/(\bar{x} - \bar{y})$. Recall that $\rho_{\text{gin}}^{N,1}(y) = \rho_{\text{gin}}^{N,1}(|y|)$. Then

$$\begin{aligned}
\int_{S_{s\infty}} \frac{x-y}{|x-y|^2} \rho_{\text{gin}}^{N,1}(y)dy &= \int_{S_{s\infty}} \frac{1}{\bar{x}-\bar{y}} \rho_{\text{gin}}^{N,1}(|y|)dy \\
&= -\int_{S_{s\infty}} \sum_{m=0}^{\infty} \bar{x}^m (\frac{1}{\bar{y}})^{m+1} \rho_{\text{gin}}^{N,1}(|y|)dy \quad \text{by } \frac{|\bar{x}|}{|\bar{y}|} < \frac{r}{s} < 1 \\
&= -\sum_{m=0}^{\infty} \bar{x}^m \int_{S_{s\infty}} \frac{y^{m+1}}{|y|^{2(m+1)}} \rho_{\text{gin}}^{N,1}(|y|)dy = 0,
\end{aligned}$$

which implies (6.10). Here we used $1/\bar{y} = y/|y|^2$ and (6.4) for the last line. $\square$

Let $S_{s\infty}^x = \{s \le |x-y| < \infty\}$. Note that $S_{s\infty} = S_{s\infty}^0$. We set

$$T_s^x = S_{s\infty} \backslash S_{s\infty}^x, \quad U_s^x = S_{s\infty}^x \backslash S_{s\infty}. \tag{6.11}$$

Lemma 6.5. (5.4) *holds with $w(x) = 2x$.*

Proof. By (4.27), $v^N(x,y) = 0$, and $g^N(x,y) = 2(x-y)/|x-y|^2$, we have

$$\begin{aligned}
\mathrm{E}^{\mu^N}[w_s^N(x,\mathsf{y})] &= \int_{S_{s\infty}^x} \frac{2(x-y)}{|x-y|^2} \rho_{\text{gin}}^{N,1}(y)dy \qquad (6.12)\\
&= -\int_{T_s^x} \frac{2(x-y)}{|x-y|^2} \rho_{\text{gin}}^{N,1}(y)dy + \int_{U_s^x} \frac{2(x-y)}{|x-y|^2} \rho_{\text{gin}}^{N,1}(y)dy \\
&\to -\int_{T_s^x} \frac{2(x-y)}{|x-y|^2} \frac{1}{\pi} dy + \int_{U_s^x} \frac{2(x-y)}{|x-y|^2} \frac{1}{\pi} dy.
\end{aligned}$$

uniformly in $x \in S_r$ as $N \to \infty$. We used here (6.10) and (6.11) for the second line, and (6.4) for the third one. By a direct calculation we obtain

$$\lim_{s\to\infty} \sup_{x \in S_r} |-\int_{T_s^x} \frac{2(x-y)}{|x-y|^2} \frac{1}{\pi} dy + \int_{U_s^x} \frac{2(x-y)}{|x-y|^2} \frac{1}{\pi} dy - 2x| = 0. \tag{6.13}$$

Combining (6.12) and (6.13) we obtain (5.4). $\square$

Lemma 6.6. *Let h_{rs} be the function on $S \times \mathsf{S}$ defined by*

$$\mathsf{h}_{rs}(x,\mathsf{y}) = \sum_{r \le |x-y_i| < s} \frac{2(x-y_i)}{|x-y_i|^2} \lceil |x-y_i| \rceil, \tag{6.14}$$

where $\lceil \cdot \rceil$ is the minimal integer greater than or equal to $\cdot$ and $\mathsf{y} = \sum_i \delta_{y_i}$. Then

$$\sup_{N\in\mathbb{N}} \sup_{x \in S_r} \mathrm{Var}^{\mu_{\text{gin}}^N}[\mathsf{h}_{rs}(x,\mathsf{y})] = O(s) \quad \text{for each } r > 0. \tag{6.15}$$

Here $f(s) = O(s)$ means $\limsup_{s\to\infty} |f(s)|/s < \infty$.

Proof. Let $S_{rs} = S_s \backslash S_r$. Let $h_{rs}(z) = 1_{S_{rs}}(z) 2z\lceil |z| \rceil/|z|^2$. Then $\mathsf{h}_{rs}(x, \mathsf{y}) = \sum_i h_{rs}(x - y_i)$ by (6.14). By a standard calculation of determinantal random point fields, we deduce that

$$\begin{aligned}\mathrm{Var}^{\mu_{\mathrm{gin}}^N}[\mathsf{h}_{rs}(x, \mathsf{y})] = & -\int_{S_{rs}^2} (h_{rs}(x-y), h_{rs}(x-z))_{\mathbb{R}^2} |\mathsf{K}_{\mathrm{gin}}^N(y,z)|^2 dydz \\ & + \int_{S_{rs}} |h_{rs}(x-y)|^2 \mathsf{K}_{\mathrm{gin}}^N(y,y) dy.\end{aligned} \tag{6.16}$$

We set $S_{rs}^x = \{r \le |x-y| < s\}$. By a direct calculation we have

$$\begin{aligned} & 1_{S_{rs}^x \cap S_{rs}}(y) |h_{rs}(x-y) - h_{rs}(-y)|/2 \\ = & 1_{S_{rs}^x \cap S_{rs}}(y) \left| \frac{x-y}{|x-y|^2} \lceil |x-y| \rceil + \frac{y}{|y|^2} \lceil |y| \rceil \right| \\ = & 1_{S_{rs}^x \cap S_{rs}}(y) \left| \{ \frac{x-y}{|x-y|^2} + \frac{y}{|y|^2} \} \lceil |x-y| \rceil - \frac{y}{|y|^2} \{ \lceil |x-y| \rceil - \lceil |y| \rceil \} \right| \\ \le & c_{10} 1_{S_{rs}^x \cap S_{rs}}(y)/|y|. \end{aligned} \tag{6.17}$$

Here $c_{10} = c_{10}(r)$ is the finite constant defined by

$$c_{10} = \sup_{\substack{x \in S_r \\ y \in S_{r\infty}^x \cap S_{r\infty}}} \left| \{ \frac{x-y}{|x-y|^2} + \frac{y}{|y|^2} \} \lceil |x-y| \rceil - \frac{y}{|y|^2} \{ \lceil |x-y| \rceil - \lceil |y| \rceil \} \right| |y|.$$

Let $c_{11} = \max\{2c_{10} + 2(r+1)/r\}$. Then by $|h_{rs}(z)| \le 1_{S_{rs}}(z) \cdot 2(r+1)/r$ and (6.17), we deduce that for all $x \in S_r$ and $y \in S$

$$|h_{rs}(x-y) - h_{rs}(-y)| \le c_{11} \{ \frac{1_{S_{rs}^x \cap S_{rs}}(y)}{|y|} + 1_{S_{rs}^x \backslash S_{rs}}(y) + 1_{S_{rs} \backslash S_{rs}^x}(y) \}. \tag{6.18}$$

By (6.3), (6.16), and (6.18) we easily deduce that

$$\sup_{N \in \mathbb{N}} \sup_{x \in S_r} |\mathrm{Var}^{\mu_{\mathrm{gin}}^N}[\mathsf{h}_{rs}(x, \mathsf{y})] - \mathrm{Var}^{\mu_{\mathrm{gin}}^N}[\mathsf{h}_{rs}(0, \mathsf{y})]| = O(s). \tag{6.19}$$

By applying [13, Lemma 9.2] to $\mathsf{h}_{rs}(0, \mathsf{y})$ we have

$$\sup_{N \in \mathbb{N}} \mathrm{Var}^{\mu_{\mathrm{gin}}^N}[\mathsf{h}_{rs}(0, \mathsf{y})] = O(s). \tag{6.20}$$

Hence (6.15) follows immediately from (6.19) and (6.20). □

Let $\mathsf{g}_{rs}(x, \mathsf{y}) = \sum_{r \le |x-y_i| < s} 2(x-y_i)/|x-y_i|^2$. Then we easily deduce that

$$\mathsf{g}_{rs} = \frac{\mathsf{h}_{rs}}{s} + \sum_{t=r+1}^{s-1} \frac{\mathsf{h}_{rt}}{t(t+1)}. \tag{6.21}$$

Lemma 6.7. *$\mathsf{g}_{r\infty}(x, \cdot) = \lim_{s\to\infty} \mathsf{g}_{rs}(x, \cdot)$ exists in $L^2(\mu_{\mathrm{gin}}^N)$ for all x and*

$$\mathsf{g}_{r\infty}(x, \cdot) = \sum_{t=r+1}^{\infty} \frac{\mathsf{h}_{rt}(x, \cdot)}{t(t+1)} \quad \textit{in } L^2(\mu_{\mathrm{gin}}^N) \quad \textit{for all } x. \tag{6.22}$$

Proof. By $\mu_{\mathrm{gin}}^N(\mathsf{s}(S)=N)=1$ we see that $\lim_{s\to\infty}\mathsf{g}_{rs}(x,\cdot)$ and $\lim_{s\to\infty}\mathsf{h}_{rs}(x,\cdot)$ exist in $L^2(\mu_{\mathrm{gin}}^N)$ for all x. Hence $\lim_{s\to\infty}\mathsf{h}_{rs}/s=0$ in $L^2(\mu_{\mathrm{gin}}^N)$ for all x. This together with (6.21) implies (6.22). □

Lemma 6.8. (5.5) *holds.*

Proof. Let $\hat{g}_{s\infty}=\sup_{N\in\mathbb{N}}\sup_{x\in S_r}\mathrm{Var}^{\mu_{\mathrm{gin}}^N}[\mathsf{g}_{s\infty}(x,\mathsf{y})]^{1/2}$. Let $\hat{h}_{rs}$ be defined similarly to $\hat{g}_{s\infty}$ by replacing $\mathsf{g}_{s\infty}$ by h_{rs}. By (6.14) we see that $\mathsf{h}_{st}=-\mathsf{h}_{rs}+\mathsf{h}_{rt}$. So $\hat{h}_{st}\le\hat{h}_{rs}+\hat{h}_{rt}$. Hence, by (6.22) we deduce that

$$\hat{g}_{s\infty}\le\sum_{t=s+1}^{\infty}\frac{\hat{h}_{st}}{t(t+1)}\le\sum_{t=s+1}^{\infty}\frac{\hat{h}_{rs}}{t(t+1)}+\sum_{t=s+1}^{\infty}\frac{\hat{h}_{rt}}{t(t+1)}. \tag{6.23}$$

By (6.15) we deduce that $\hat{h}_{rs}=O(\sqrt{s})$. Combining this with (6.23) we deduce that $\lim_{s\to\infty}\hat{g}_{s\infty}=0$, which yields (5.5). □

Proof of Theorem 6.1. We use Theorem 4.5 to prove Theorem 6.1. So we check that $\{\mu_{\mathrm{gin}}^N\}$ satisfies the conditions in Theorem 4.5. By Lemma 6.2 it only remains to prove (4.30). Recall that (4.30) follows from (5.2)–(5.5). We obtain (5.2) and (5.3) by Lemma 6.3. (5.4) follows from Lemma 6.5. (5.5) follows from Lemma 6.8. □

7 Proof of Theorems 2.1–2.3.

In this section we prove Theorems 2.1–2.3. We recall that we took $v(x,y)=0$ in Lemma 6.2. So we set

$$\mathsf{g}_{rs}(x,\mathsf{y})=\sum_{r\le|x-y_i|<s}2(x-y_i)/|x-y_i|^2,\quad \mathsf{g}_s=\mathsf{g}_{0s}, \tag{7.1}$$

$$\tilde{\mathsf{g}}_{rs}(x,\mathsf{y})=\sum_{r\le|y_i|<s}2(x-y_i)/|x-y_i|^2,\quad \tilde{\mathsf{g}}_s=\tilde{\mathsf{g}}_{0s}, \tag{7.2}$$

where $x\in S$ and $\mathsf{y}=\sum_i\delta_{y_i}$.

Lemma 7.1.

$$\lim_{s\to\infty}\{\mathsf{g}_s(x,\cdot)-\tilde{\mathsf{g}}_s(x,\cdot)\}=-2x\ \textit{ in } L^2(\mu_{\mathrm{gin}})\ \textit{compact uniformly in } x. \tag{7.3}$$

Proof. Let T_s^x and U_s^x be as in (6.11). By $\rho_{\mathrm{gin}}^1(x)=1/\pi$ and (6.13) we deduce that

$$\begin{aligned}&\lim_{s\to\infty}\mathrm{E}^{\mu_{\mathrm{gin}}}[\mathsf{g}_s(x,\mathsf{y})-\tilde{\mathsf{g}}_s(x,\mathsf{y})] \qquad (7.4)\\ &=\lim_{s\to\infty}\{\int_{T_s^x}\frac{2(x-y)}{|x-y|^2}\rho_{\mathrm{gin}}^1(y)dy-\int_{U_s^x}\frac{2(x-y)}{|x-y|^2}\rho_{\mathrm{gin}}^1(y)dy\}\\ &=-2x\quad\text{compact uniformly in } x.\end{aligned}$$

By a similar equality to (6.16) with $\mathsf{K}_{\mathrm{gin}}(y,y)=1/\pi$ we deduce that

$$\begin{aligned}&\lim_{s\to\infty}\mathrm{Var}^{\mu_{\mathrm{gin}}}[\mathsf{g}_s(x,\mathsf{y})-\tilde{\mathsf{g}}_s(x,\mathsf{y})] \qquad (7.5)\\ &\le\lim_{s\to\infty}\{\int_{T_s^x}\frac{4}{|x-y|^2}\frac{1}{\pi}dy+\int_{U_s^x}\frac{4}{|x-y|^2}\frac{1}{\pi}dy\}\\ &=0\quad\text{compact uniformly in } x.\end{aligned}$$

By (7.4) and (7.5) we obtain (7.3). $\square$

We next prove the identity of the form

$$\lim_{s\to\infty} \mathsf{g}_s(x,\mathsf{y}) = -2x + \lim_{s\to\infty} \tilde{\mathsf{g}}_s(x,\mathsf{y}). \tag{7.6}$$

Lemma 7.2. (1) *For all $x \in S_r$, $\mathsf{g}_{rs}(x,\mathsf{y})$ and $\tilde{\mathsf{g}}_{rs}(x,\mathsf{y})$ converge in $L^2(\mu_{\mathrm{gin}})$ as $s \to \infty$ compact uniformly in $x \in S_r$. Moreover,* (7.6) *holds in the sense that*

$$\mathsf{g}_r(x,\mathsf{y}) + \lim_{s\to\infty} \mathsf{g}_{rs}(x,\mathsf{y}) = -2x + \tilde{\mathsf{g}}_r(x,\mathsf{y}) + \lim_{s\to\infty} \tilde{\mathsf{g}}_{rs}(x,\mathsf{y}). \tag{7.7}$$

(2) *For all x, $\mathsf{g}_s(x,\mathsf{y})$ and $\tilde{\mathsf{g}}_s(x,\mathsf{y})$ converge in $L^2(\mu_{\mathrm{gin},x})$ as $s \to \infty$ compact uniformly in x. Moreover,* (7.6) *holds.*
(3) g_s *and* $\tilde{\mathsf{g}}_s$ *converge in $L^2_{\mathrm{loc}}(\mu^1_{\mathrm{gin}})$ as $s \to \infty$ and* (7.6) *holds.*

Remark 7.1. Note that $\mathsf{g}_r(x,\cdot), \tilde{\mathsf{g}}_r(x,\cdot) \notin L^2(\mu_{\mathrm{gin}})$ because of the singularity at x. So the statement of (1) is weaker than the others.

Proof. By [15, Theorem 1.3] we see that $\mathrm{Var}^{\mu_{\mathrm{gin}}}[\mathsf{h}_{rs}(x,\mathsf{y})] = O(s)$ compact uniformly in $x \in S_r$. Since $\mathrm{E}^{\mu_{\mathrm{gin}}}[\mathsf{h}_{rs}(x,\mathsf{y})] = 0$ for all $x \in S_r$, we deduce that $\mathrm{E}^{\mu_{\mathrm{gin}}}[|\mathsf{h}_{rs}(x,\mathsf{y})|^2] = \mathrm{Var}^{\mu_{\mathrm{gin}}}[\mathsf{h}_{rs}(x,\mathsf{y})] = O(s)$ compact uniformly in $x \in S_r$. Hence $\lim_{s\to\infty} \mathsf{h}_{rs}/s = 0$ in $L^2(\mu_{\mathrm{gin}})$ compact uniformly in $x \in S_r$. From this, combined with (6.21), we deduce that $\mathsf{g}_{r\infty} := \lim_{s\to\infty} \mathsf{g}_{rs}$ converges in $L^2(\mu_{\mathrm{gin}})$ compact uniformly in $x \in S_r$. So by (7.3) we obtain (7.7). We have thus proved (1).

By (6.9) and (6.1) and a similar representation of correlation functions of $\mu^N_{\mathrm{gin},x}$ we deduce that the first statement of (2) follows from that of (1). Since $\mathsf{g}_r, \tilde{\mathsf{g}}_r \in L^2(\mu_{\mathrm{gin},x})$, the second follows from (7.7). So we obtain (2).

(3) follows from (2) and the relation $\mu^1_{\mathrm{gin}}(A\times B) = \int_A \mu_{\mathrm{gin},x}(B)\rho^1_{\mathrm{gin}}(x)dx$ with $\rho^1_{\mathrm{gin}}(x) = 1/\pi$. $\square$

Proof of Theorems 2.1 and 2.3. We use Theorems 2.6 and 2.7 to prove Theorem 2.1. We take $\mu = \mu_{\mathrm{gin}}$ and $\mathsf{b}(x,\mathsf{y}) = (1/2)\lim_s \mathsf{g}_s(x,\mathsf{y})$, where g_s is same as (7.6). Moreover, $\sigma(x,\mathsf{y})$ is the unit matrix for all (x,y). Hence $\mathsf{a} = \sigma^2$ is also the unit matrix. We check that μ_{gin} satisfies (A.1)–(A.5) for these σ and b.

(A.1) and (A.5) are clear from (2.4) and (2.5). (A.2) follows from Theorem 6.1 and Lemma 7.2 (3). In [13, Theorem 2.6] we proved that the closability in (A.3) holds for $k = 0$. Indeed, we proved that μ_{gin} is a quasi-Gibbs measure in the sense of [13, Definition 2.1] and deduced the closability for $k = 0$ from this. The closability for general $k \in \mathbb{N}$ also follows in a similar fashion from the quasi-Gibbs property of μ_{gin}. Since the kernel $\mathsf{K}_{\mathrm{gin}}$ is locally Lipschitz continuous, (A.4) immediately follows from [12, Theorem 2.1].

We thus see that μ_{gin} satisfies (A.1)–(A.5). Hence Theorems 2.1 and 2.3 follow from Theorems 2.6 and 2.7, respectively. $\square$

Proof of Theorem 2.2. By Lemma 7.2 (3) we see that (7.6) holds in $L^2_{\mathrm{loc}}(\mu^1_{\mathrm{gin}})$. Hence we deduce that $\mathsf{b}(x,\mathsf{y}) = -x + \tilde{\mathsf{b}}(x,\mathsf{y})$ in $L^2_{\mathrm{loc}}(\mu^1_{\mathrm{gin}})$. This combined with Theorem 2.1 implies Theorem 2.2. $\square$

8 Proof of Theorems 2.4 and 2.5.

In this section we prove Theorems 2.4 and 2.5 by using Theorems 2.6 and 2.7. So we take $\mu = \mu_{\mathrm{dys},\beta}$ and prove that $\mu_{\mathrm{dys},\beta}$ satisfies (A.1)–(A.5).

Lemma 8.1. *$\mu_{\mathrm{dys},\beta}$ ($\beta = 1, 2, 4$) satisfy* (A.1)*,* (A.3)*,* (A.4)*, and* (A.5)*.*

Proof. Since the correlation functions $\{\rho_\beta^n\}$ of $\mu_{\mathrm{dys},\beta}$ have the expression (2.17) and the kernels K_β are bounded, (A.1) and (A.5) are clear.

In [13, Theorem 2.5] we proved that the closability in (A.3) holds for $k = 0$. Indeed, we proved that $\mu_{\mathrm{dys},\beta}$ is a quasi-Gibbs measure and deduced the closability for $k = 0$ from this. The closability for general $k \geq 1$ also follows from the quasi-Gibbs property of $\mu_{\mathrm{dys},\beta}$ in a similar fashion. Since the kernel K_β is locally Lipschitz continuous, (A.4) follows from [12, Theorem 2.1]. $\square$

By Lemma 8.1 it only remains to prove (A.2). Define $\mathsf{K}_\beta^N(x)$ by (9.4)–(9.6) with the replacement of $S(x)$ by $S_N(x) = \sin(\pi x)/\{N\sin(\pi x/N)\}$. We set $R_N = (-N/2, N/2]$ and

$$\mathsf{K}_\beta^N(x,y) = 1_{R_N}(x)\mathsf{K}_\beta^N(x-y)1_{R_N}(y).$$

We take μ^N in (A.2) to be the probability measure μ_β^N on S whose n-point correlation function $\rho_\beta^{N,n}$ is given by

$$\rho_\beta^{N,n}(\mathbf{x}) = \det[\mathsf{K}_\beta^N(x_i, x_j)]_{1\leq i,j\leq n}, \tag{8.1}$$

where $\mathbf{x} = (x_i)$. It is well known [8] that $\mu_\beta^N(\mathsf{s}(\mathbb{R}) = N) = 1$ and that

$$\rho_\beta^{N,N}(\mathbf{x}) = \text{const.} \prod_{i,j=1,i<j}^{N} 1_{R_N}(x_i)|e^{2\pi \mathrm{i} x_i/N} - e^{2\pi \mathrm{i} x_j/N}|^\beta 1_{R_N}(x_j). \tag{8.2}$$

We can regard R_N as a torus and μ_β^N to be a translation invariant probability measure on the configuration space on the torus R_N. The image measure of μ_β^N under the map $\omega_N(x) = e^{2\pi \mathrm{i} x/N}$ gives the distributions of the eigenvalues of the random matrices called circular ensembles [8]. We can rewrite (8.2) as

$$\rho_\beta^{N,N}(\mathbf{x}) = \text{const.} \prod_{i,j=1,i<j}^{N} 1_{R_N}(x_i)|\omega_N(x_i) - \omega_N(x_j)|^\beta 1_{R_N}(x_j). \tag{8.3}$$

Taking (8.2) into consideration we set

$$\begin{aligned} g^N(x,y) &= \frac{\partial}{\partial x}\log|e^{2\pi \mathrm{i} x/N} - e^{2\pi \mathrm{i} y/N}|^\beta &\quad& \text{if } x,y \in (-N/2, N/2) \\ &= 0 && \text{otherwise.} \end{aligned} \tag{8.4}$$

Then we can easily check that

$$\limsup_{N\to\infty} \sup_{x\in S_r} |\int_{S_{s\infty}^x} g^N(x,y)dy| = o(s) \quad (s \to \infty) \tag{8.5}$$

and that there exists a constant c_{12} such that

$$\sup_{N\geq 8r} \sup_{x\in S_r} |g^N(x,y)| \leq c_{12}\min\{1, 1/|y|\} \quad \text{for all } |y| > 2r. \tag{8.6}$$

Theorem 8.2. *Suppose $\beta = 1, 2, 4$. Then the log derivative $\mathsf{d}^{\mu_{\mathrm{dys},\beta}}$ exists in $L^p_{\mathrm{loc}}(\mu^1_{\mathrm{dys},\beta})$ for any $1 < p < 2$. Moreover $\mathsf{d}^{\mu_{\mathrm{dys},\beta}}$ is given by*

$$\mathsf{d}^{\mu_{\mathrm{dys},\beta}}(x, \mathsf{y}) = \lim_{r\to\infty} \sum_{|x-y_i|<r} \frac{\beta}{x - y_i} \qquad (\mathsf{y} = \sum_i \delta_{y_i}). \tag{8.7}$$

Proof. We use Theorem 4.5 to prove Theorem 8.2. So we check the conditions of Theorem 4.5. We take $u(x) = w(x) = 0$, $u^N(x) = \delta_{-N/2}(x) - \delta_{N/2}(x)$, where $\delta_{\pm N/2}(x)$ are delta measures, and $v^N(x, y) = v(x, y) = 0$. We set g^N as (8.4) and $g(x, y) = 2/(x - y)$.

The conditions (4.1) and (4.2) follow from (8.1) and the definition of K^N_β. (4.14) and (4.28) are clear. For $\beta = 2, 4$, the condition (4.29) with $\hat{p} = 2$ follows from (2.17), $g(x, y) = 2/(x-y)$, (8.4), and Lemma 5.1. For $\beta = 1$ one can check that (4.29) with $1 < \hat{p} < 2$ holds by the Hölder inequality in addition to the above.

We next prove (4.30). For this it is sufficient to check (5.7)–(5.10) by Lemma 5.3. Let $\mu^N_{\beta,x}$ be the Palm measure of μ^N_β conditioned at $x \in R_N$ and let $\rho^{N,n}_{\beta,x}$ be its n-point correlation function. Then $\mu^N_{\beta,x}$ has a determinantal structure with kernel

$$\mathsf{K}^N_{\beta,x}(y, z) = \mathsf{K}^N_\beta(y, z) - \mathsf{K}^N_\beta(y, x)\mathsf{K}^N_\beta(x, z)/\mathsf{K}^N_\beta(x, x). \tag{8.8}$$

When $\beta = 2$, (8.8) follows from [18, Theorem 1.7]. When $\beta = 1, 4$, one can also check (8.8). By (9.2) and (9.4)–(9.6) we easily see that $\mathsf{K}^N_\beta(x, x) = 1_{R_N}(x)$. Hence (8.8) implies that for $x \in R_N$ and $y, z \in \mathbb{R}$

$$\mathsf{K}^N_{\beta,x}(y, z) = \mathsf{K}^N_\beta(y, z) - \mathsf{K}^N_\beta(y, x)\mathsf{K}^N_\beta(x, z). \tag{8.9}$$

By (8.1) and (8.9) we see that for $x \in R_N$ and $y, z \in \mathbb{R}$

$$\rho^{N,1}_{\beta,x}(y) - \rho^{N,1}_\beta(y) = -[\mathsf{K}^N_\beta(y, x)\mathsf{K}^N_\beta(x, y)]^{(0)}. \tag{8.10}$$

Here $[\cdot]^{(0)}$ means the scaler part of quaternions $\cdot$ in the sense of the Appendix. When $\beta = 2$, $[\cdot]^{(0)} = \cdot$ because $\cdot$ are complex numbers. By (8.10), (9.7) and (9.8) there exists a constant c_{13} satisfying

$$\sup_{N\geq 8r} \sup_{x\in S_r} |\rho^{N,1}_{\beta,x}(y) - \rho^{N,1}_\beta(y)| \leq c_{13} \min\{1, 1/|y|\} \quad \text{for all } |y| > 2r. \tag{8.11}$$

By (8.6) and (8.11) we obtain (5.7).

By (8.1) and (8.9) we see that for $x \in R_N$ and $y, z \in \mathbb{R}$

$$\begin{aligned} &\rho^{N,2}_{\beta,x}(y, z) - \rho^{N,2}_\beta(y, z) \\ &= - [\mathsf{K}^N_\beta(y, x)\mathsf{K}^N_\beta(x, y)]^{(0)} - [\mathsf{K}^N_\beta(z, x)\mathsf{K}^N_\beta(x, z)]^{(0)} \\ &\quad + [\mathsf{K}^N_\beta(y, x)\mathsf{K}^N_\beta(x, z)\mathsf{K}^N_\beta(z, y)]^{(0)} + [\mathsf{K}^N_\beta(y, z)\mathsf{K}^N_\beta(z, x)\mathsf{K}^N_\beta(x, y)]^{(0)}. \end{aligned} \tag{8.12}$$

Then by (8.5), (8.6), (9.7) and (9.8) we see that as $s \to \infty$

$$\begin{aligned} &\limsup_{N\to\infty} \sup_{x\in S_r} \left| \int_{(S^x_{s\infty})^2} g^N(x, y) g^N(x, z) [\mathsf{K}^N_\beta(y, x)\mathsf{K}^N_\beta(x, y)]^{(0)} dydz \right| \\ &\leq o(s) \cdot \limsup_{N\to\infty} \sup_{x\in S_r} \int_{S^x_{s\infty}} \left| g^N(x, y) [\mathsf{K}^N_\beta(y, x)\mathsf{K}^N_\beta(x, y)]^{(0)} \right| dy \quad \text{by (8.5)} \\ &= o(s) \qquad\qquad \text{by (8.6), (9.7), and (9.8).} \end{aligned} \tag{8.13}$$

By (8.6), (9.7) and (9.9) we deduce that

$$\limsup_{N\to\infty} \sup_{x\in S_r} \int_{(S_{s\infty}^x)^2} |g^N(x,y)g^N(x,z)[\mathsf{K}_\beta^N(y,x)\mathsf{K}_\beta^N(x,z)\mathsf{K}_\beta^N(z,y)]^{(0)}|dydz$$
$$= o(s) \quad (s\to\infty). \tag{8.14}$$

We therefore obtain (5.8) from (8.12), (8.13) and (8.14).

By $\rho_\beta^{N,1} = 1_{R_N}$ and (8.5) we obtain (5.9) because $v^N(x,y) = w(x) = 0$.

By (8.1) and $\mathsf{K}_\beta^N(y,y) = 1_{R_N}(y)$ we deduce that

$$\rho_\beta^{N,2}(y,z) = 1_{R_N}(y)1_{R_N}(z)\{1-[\mathsf{K}_\beta^N(y,z)\mathsf{K}_\beta^N(z,y)]^{(0)}\}. \tag{8.15}$$

So we deduce from (8.5), (9.7), (9.8), and (8.15) that

$$\limsup_{N\to\infty} \sup_{x\in S_r} |\int_{(S_{s\infty}^x)^2} g^N(x,y)g^N(x,z)\rho_\beta^{N,2}(y,z)dydz| \tag{8.16}$$
$$= o(s) + \limsup_{N\to\infty} \sup_{x\in S_r} |\int_{(S_{s\infty}^x)^2} g^N(x,y)g^N(x,z)[\mathsf{K}_\beta^N(y,z)\mathsf{K}_\beta^N(z,y)]^{(0)}dydz|$$
$$= o(s) \quad (s\to\infty).$$

Since $\rho_\beta^{N,1}(y) = 1_{R_N}(y)$, we deduce from (8.6) that

$$\limsup_{N\to\infty} \sup_{x\in S_r} \int_{S_{s\infty}^x} |g^N(x,y)|^2 \rho_\beta^{N,1}(y)dy = \ o(s) \quad (s\to\infty). \tag{8.17}$$

Hence by (8.16) and (8.17) we obtain (5.10). □

Lemma 8.3. *Suppose $\beta = 2,4$. Then $\mathsf{d}^{\mu_{\mathrm{dys},\beta}} \in L^2_{\mathrm{loc}}(\mu^1_{\mathrm{dys},\beta})$.*

Proof. Let $\mathsf{g}_s(x,\mathsf{y}) = \sum_{|x-y_i|<s} 2/(x-y_i)$. Then by Theorem 8.2 it is sufficient for Lemma 8.3 to prove g_s converge in $L^2_{\mathrm{loc}}(\mu^1_{\mathrm{dys},\beta})$. Let μ_x be the Palm measure of $\mu_{\mathrm{dys},\beta}$ conditioned at x. Then since $\mu_{\mathrm{dys},\beta}$ are translation invariant, it is enough to show that $\mathsf{g}_s(x,\mathsf{y})$ converge in $L^2(\mu_x)$ for each x.

Let $\mathsf{h}_s(x,\mathsf{y}) = \sum_{|x-y_i|<s} 2\lceil |x-y_i|\rceil/(x-y_i)$. Then we see that

$$\mathsf{g}_s = \frac{\mathsf{h}_s}{s} + \sum_{t=1}^{s-1} \frac{\mathsf{h}_t}{t(t+1)}. \tag{8.18}$$

By the calculation based on the 1 and 2-point correlation functions we can check $\mathrm{E}^{\mu_x}[|\mathsf{h}_s|^2] \sim O(s)$. This combined with (8.18) completes the proof. □

Proof of Theorems 2.4 and 2.5. By Lemma 8.1, Theorem 8.2, and Lemma 8.3 we see that $\mu_{\mathrm{dys},\beta}$ ($\beta = 2,4$) satisfy (A.1)–(A.5). Hence Theorems 2.4 and 2.5 follow from Theorems 2.6 and 2.7, respectively. When $\beta = 1$, $\mathsf{d}^{\mu_{\mathrm{dys},\beta}} \in L^p_{\mathrm{loc}}(\mu^1_{\mathrm{dys},\beta})$ for any $1 < p < 2$ and $\mathsf{d}^{\mu_{\mathrm{dys},\beta}} \notin L^2_{\mathrm{loc}}(\mu^1_{\mathrm{dys},\beta})$. In this case we can justify (2.39) by using the localization, and we still have Theorems 2.4 and 2.5. □

9 Appendix.

We begin by defining K_β for $\beta = 1, 4$. For this purpose, we recall the standard quaternion notation for 2×2 matrices (see [8, Ch. 2.4]),

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}. \tag{9.1}$$

A quaternion q is represented by $q = q^{(0)}\mathbf{1} + q^{(1)}\mathbf{e}_1 + q^{(2)}\mathbf{e}_2 + q^{(3)}\mathbf{e}_3$, where $q^{(i)}$ are complex numbers. There is a natural identification between the 2×2 complex matrices and the quaternions given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}(a+d)\mathbf{1} - \frac{\mathrm{i}}{2}(a-d)\mathbf{e}_1 + \frac{1}{2}(b-c)\mathbf{e}_2 - \frac{\mathrm{i}}{2}(b+c)\mathbf{e}_3. \tag{9.2}$$

We denote by $\Theta(\begin{bmatrix} a & b \\ c & d \end{bmatrix})$ the quaternion defined by the right hand side of (9.2).

For a quaternion $q = q^{(0)}\mathbf{1} + q^{(1)}\mathbf{e}_1 + q^{(2)}\mathbf{e}_2 + q^{(3)}\mathbf{e}_3$, we call $q^{(0)}$ the scalar part of q. A quaternion is called scalar if $q^{(i)} = 0$ for $i = 1, 2, 3$. We often identify a scalar quaternion $q = q^{(0)}\mathbf{1}$ with the complex number $q^{(0)}$.

Let $\bar{q} = q^{(0)}\mathbf{1} - \{q^{(1)}\mathbf{e}_1 + q^{(2)}\mathbf{e}_2 + q^{(3)}\mathbf{e}_3\}$. A quaternion matrix $A = [a_{ij}]$ is called self-dual if $a_{ij} = \bar{a}_{ji}$ for all i, j. For a self-dual $n \times n$ quaternion matrix $A = [a_{ij}]$ we set

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}[\sigma] \prod_{i=1}^{L(\sigma)} [a_{\sigma_i(1)\sigma_i(2)} a_{\sigma_i(2)\sigma_i(3)} \cdots a_{\sigma_i(\ell-1)\sigma_i(\ell)}]^{(0)}. \tag{9.3}$$

Here $\sigma = \sigma_1 \cdots \sigma_{L(\sigma)}$ is a decomposition of σ to products of the cyclic permutations $\{\sigma_i\}$ with disjoint indices. We write $\sigma_i = (\sigma_i(1), \sigma_i(2), \ldots, \sigma_i(\ell))$, where ℓ is the length of the cyclic permutation σ_i. The decomposition is unique up to the order of $\{\sigma_i\}$. As before $[\cdot]^{(0)}$ means the scalar part of the quaternion $\cdot$. It is known that the right hand side is well defined (see [8, Section 5.1]).

We are now ready to introduce K_β. Let $S(x) = \sin(\pi x)/\pi x$ and define

$$\mathsf{K}_1(x) = \Theta(\begin{bmatrix} S(x) & \frac{dS}{dx}(x) \\ \int_0^x S(y)dy - \frac{1}{2}\mathrm{sgn}(x) & S(x) \end{bmatrix}), \tag{9.4}$$

$$\mathsf{K}_2(x) = S(x), \tag{9.5}$$

$$\mathsf{K}_4(x) = \Theta(\begin{bmatrix} S(2x) & \frac{dS}{dx}(2x) \\ \int_0^{2x} S(y)dy & S(2x) \end{bmatrix}). \tag{9.6}$$

We thus clarify the meaning of (2.17).

We set the kernels K_β^N by (9.4)–(9.6) with the replacement of $S(x)$ by $S_N(x) = \sin(\pi x)/\{N \sin(\pi x/N)\}$. Let $\omega_N(x) = e^{2\pi \mathrm{i} x/N}$ as before, and set

$$\eta_N^{x,y} = 1_{R_N}(x) 1_{R_N}(y) \min\{1, 1/|\omega_N(x) - \omega_N(y)|\}. \tag{9.7}$$

Then by (9.2) and (9.4)–(9.6) there exist constants c_{14} and c_{15} such that

$$|[\mathsf{K}_\beta^N(x,y)\mathsf{K}_\beta^N(y,x)]^{(0)}| \le c_{14}\eta_N^{x,y}, \tag{9.8}$$

$$|[\mathsf{K}_\beta^N(x,y)\mathsf{K}_\beta^N(y,z)\mathsf{K}_\beta^N(z,x)]^{(0)}| \le c_{15}\{\eta_N^{x,y}\eta_N^{y,z} + \eta_N^{y,z}\eta_N^{z,x} + \eta_N^{z,x}\eta_N^{x,y}\}. \tag{9.9}$$

References

[1] Forrester, Peter J. , *Log-gases and Random Matrices*, London Mathematical Society Monographs (in press).

[2] Fukushima, M., Oshima, Y., Takeda, M., *Dirichlet forms and symmetric Markov processes*, Walter de Gruyter (1994).

[3] Fritz, J. *Gradient dynamics of infinite point systems*, Ann. Probab. **15** (1987) 478-514.

[4] Inukai, K., *Collision or non-collision problem for interacting Brownian particles*,, Proc. Japan Acad. Ser. A Math. Sci. **82**, (2006), 66-70.

[5] Katori, M., Tanemura, H., *Non-equilibrium dynamics of Dyson's model with an infinite number of particles*, published on line in Commun. Math. Phys..

[6] Lang, R., *Unendlich-dimensionale Wienerprocesse mit Wechselwirkung I*, Z. Wahrschverw. Gebiete **38** (1977) 55-72.

[7] Lang, R., *Unendlich-dimensionale Wienerprocesse mit Wechselwirkung II*, Z. Wahrschverw. Gebiete **39** (1978) 277-299.

[8] Mehta, M., *Random matrices*, (Third Edition) Elsevier 2004.

[9] Nagao T., Forrester, P.J., *Multilevel dynamical correlation functions for Dyson's Brownian motion model of random matrices*, Phys. Lett. **A247**, (1998) 42-46.

[10] Osada, H., *Dirichlet form approach to infinitely dimensional Wiener processes with singular interactions*, Commun. Math. Physic. (1996), 117-131.

[11] Osada, H. *Positivity of the self-diffusion matrix of interacting Brownian particles with hard core*, Probab. Theory Relat. Fields **112** (1998), 53–90.

[12] Osada, H., *Non-collision and collision properties of Dyson's model in infinite dimensions and other stochastic dynamics whose equilibrium states are determinantal random point fields*, in Stochastic Analysis on Large Scale Interacting Systems, eds. T. Funaki and H. Osada, Advanced Studies in Pure Mathematics **39**, 2004, 325-343.

[13] Osada, H., *Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials*, (preprint) available at "http://arxiv.org/abs/0902.3561v1" (arXiv:0902.3561v1 [math.PR]).

[14] Osada, H., *Tagged particle processes and their non-explosion criteria*, (to appear in J. Math. Soc. Japan) available at "http://arxiv.org/abs/0905.3973v3" (arXiv:0905.3973v3 [math.PR]).

[15] Osada, H. and Shirai, T., *Variance of the linear statistics of the Ginibre random point field*, in Proc. of RIMS Workshop on Stochastic Analysis and Applications, eds. M. Fukushima and I. Shigekawa, RIMS Kôkyûroku Bessatsu **B6**, (2008) 193-200.

[16] Ruelle, D., *Superstable interactions in classical statistical mechanics*, Commun. Math. Phys. **18** (1970) 127–159.

[17] Shiga, T. *A remark on infinite-dimensional Wiener processes with interactions*, Z. Wahrschverw. Gebiete **47** (1979) 299-304.

[18] Shirai, T., Takahashi, Y. *Random point fields associated with certain Fredholm determinants I: Fermion, Poisson and Boson processes*, J. Funct. Anal. **205**: (2003), 414-463.

[19] Soshnikov, A., *Determinantal random point fields*, Russian Math. Surveys **55**:5 (2000) 923-975.

[20] Spohn, H., *Interacting Brownian particles: a study of Dyson's model*, In: Hydrodynamic Behavior and Interacting Particle Systems, ed. by G.C. Papanicolaou, IMA Volumes in Mathematics **9** , Springer-Verlag (1987) 151-179.

[21] Tanemura, H., *A system of infinitely many mutually reflecting Brownian balls in* $\mathbb{R}^d$, Probab. Theory Relat. Fields **104** (1996) 399-426.